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Kähler metrics and Yukawa couplings in magnetized brane models[1]

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Abstract: Using the field theoretical approach introduced by Cremades, Ibáñez and Marchesano for describing open strings attached to D9 branes having different magnetizations, we give a procedure for determining the Kähler metrics of those open strings in toroidal compactifications.

Keywords: Flux compactifications, D-branes, Supersymmetric Effective Theories

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Contents

1 Introduction

In order to connect string theory with experimental data we need to develop techniques that allow us to compute the low-energy four-dimensional effective action that can be directly compared with what we observe in high energy experiments, starting from the original ten dimensional theory compactified on $R^{3,1} \times M_6$, where M_6 is a compact six dimensional manifold. In particular, it is important to understand how the four-dimensional physics depends on the size and the shape of M_6.

Recently, in the framework of toroidal compactifications with a number of stacks of intersecting or of their T-dual magnetized D branes, semi-realistic string extensions of the Standard Model and of the Minimal Supersymmetric Standard Model have been constructed.[1] In the magnetized brane scenario quarks and leptons correspond to open strings having their end-points attached to D branes with different magnetizations. Those open strings are called in the literature *dycharged, chiral* or *twisted* strings. Since in the case of a constant magnetization the dynamics of the open strings can be completely and analytically determined and the open strings can be exactly quantized [2], from the computation of string amplitudes one can in principle determine the low-energy four-dimensional effective Lagrangian involving those fermionic chiral strings and their supersymmetric bosonic

[1] See ref. [1] for a description of some of those models.

partners. In particular, by computing three and four-point string amplitudes, in refs. [3, 4] the dependence on the magnetization of the Kähler metrics of the twisted strings has been determined. On the other hand, using instanton calculus and the holomorphicity of the superpotential for $M_6 = T^2 \times T^2 \times T^2$, it has been seen [5–7] that the Kähler metrics of the twisted strings contain an additional explicit dependence on the moduli and also an arbitrary factor that up to now has not been possible to fix by an explicit string calculation.

However, if one is not immediately interested in the string corrections to the parameters of the low-energy four-dimensional effective action, one does not really need to compute string amplitudes, but one can directly start from the action of $\mathcal{N} = 1$ super Yang-Mills in ten dimensions with gauge group $\mathrm{U}(M)$, that describes the open strings attached to M D9 branes, perform on it a Kaluza-Klein reduction from ten to four dimensions and derive the low-energy four-dimensional effective action for the massless excitations. In particular, by imposing that the background gauge field in the six compact dimensions and along the Cartan subalgebra of $\mathrm{U}(M)$ is non-vanishing and corresponds to a constant gauge field strength, one gets a field theoretical description of the twisted open strings, previously defined. This approach pioneered in a beautiful paper by Cremades, Ibáñez and Marchesano [8] for computing the Yukawa couplings of chiral matter and extended in ref. [9] to orbifolds of toroidal models and in ref. [10] to some non-toroidal compactifications, is the one that we are going to use for computing the Kähler metrics of twisted open strings.

In this paper, following the approach of ref. [8], we give a procedure for computing the Kähler metric of twisted and untwisted scalar fields. In fact, unlike ref. [8], we do not normalize to 1 the wave-functions in the compact extra-dimensions, but instead we keep the moduli dependence that naturally comes from the integral over the compact manifold. In this way we can correctly reproduce the dependence on the moduli of the Kähler metrics apart from factors involving the magnetizations. In order to get also these factors, we add a normalization factor for each scalar field and for its supersymmetric fermionic partner that is then determined by requiring that the Yukawa couplings, that we also compute, come from a holomorphic super-potential. In this way we reproduce the Kähler metric of the adjoint scalars, of those of the hypermultiplet and of those of the chiral multiplet without additional arbitrary factors. It must also be said, however, that this is of course the minimal way to eliminate non-holomorphic factors in the Yukawa couplings. One could, in principle, also include additional factors that do not spoil the holomorphicity of the Yukawa couplings. Finally, there is also the issue of how the four-dimensional complex field depends on the ten-dimensional fields that will be discussed in the conclusions.

The paper is organized as follows. In section 2 we consider the terms of the ten-dimensional action that are relevant in our calculations and we perform the Kaluza-Klein reduction from ten to four dimensions. Using the wave functions in the extra dimensions, that are computed in appendix B, in section 3 we determine the Kähler metrics of the various scalar fields apart from a normalization factor for each four-dimensional field. In section 4 we compute the Yukawa couplings and fix the previous normalization factors by requiring that the Yukawa couplings come from a holomorphic superpotential. In section 5 we insert those normalization factors in the two-point functions for the various scalar fields and we determine their Kähler metrics showing that the expression obtained are consistent with previous string calculations in the field theory limit. Section 6 is devoted to the conclusions and to a discussion of the form of the four-dimensional scalar fields in terms of the original ten-dimensional ones.

Some appendices follow. Appendix A is devoted to a description of the torus T^2 and of the moduli used in supergravity. In appendix B we solve both the bosonic and fermionic eigenvalue equations for the wave functions in the extra dimensions obtaining the explicit wave functions. In appendix C we add few details on the calculation of the Yukawa couplings and finally in appendix D we discuss the four-dimensional supersymmetry transformations.

2 The KK reduction of the relevant terms of the action

The starting point of our analysis is the low-energy limit of the DBI action describing a set of M D9 branes, namely supersymmetric $\mathcal{N} = 1$ super Yang-Mills with gauge group U(M):

$$S = \frac{1}{g^2} \int d^{10}X \mathrm{Tr}\Big(-\frac{1}{4} F_{MN} F^{MN} + \frac{i}{2} \bar{\lambda} \Gamma^M D_M \lambda \Big), \tag{2.1}$$

where $g^2 = 4\pi e^{\phi_{10}} (2\pi\sqrt{\alpha'})^6$ and

$$F_{MN} = \nabla_M A_N - \nabla_N A_M - i[A_M, A_N]\,; \qquad D_M \lambda = \nabla_M \psi - i[A_M, \lambda] \tag{2.2}$$

being λ a ten dimensional Weyl-Majorana spinor.

We separate the generators of the gauge group into those, called U_a, that are in the Cartan subalgebra and those, called e_{ab}, outside of it [8, 10]:

$$(U_a)_{ij} = \delta_{ai}\delta_{aj}\,, \qquad (e_{ab})_{ij} = \delta_{ai}\delta_{bj} \quad (a \neq b)\,. \tag{2.3}$$

The gauge field A_M and the gaugino are expanded as

$$A_M = B_M + W_M = B_M^a U_a + W_M^{ab} e_{ab}\,; \qquad \lambda = \chi + \Psi = \chi^a U_a + \Psi^{ab} e_{ab}\,. \tag{2.4}$$

Requiring that $A_M^\dagger = A_M$ implies that B_M^a is real and $(W_M^{ab})^* = W_M^{ba}$. The same is true for the gaugino and its components χ and Ψ.

By inserting in eq. (2.1) the expansions given in eq. (2.4), we can rewrite the original action in terms of the fields B, W, χ and Ψ. Its explicit expression can be found in refs. [8, 10].

We separate the ten-dimensional coordinate X^M into a four-dimensional non-compact coordinate x^μ and a six-dimensional compact variable y^i and perform a Kaluza-Klein reduction of the Lagrangian in eq. (2.1) expanding around the background fields:

$$B_M^a(x^\mu, y^i) = \langle B_M^a \rangle (y^i) + \delta B_M^a(x^\mu, y^i)\,, \tag{2.5}$$
$$W_M^{ab}(x^\mu, y^i) = 0 + \Phi_M^{ab}(x^\mu, y^i) \tag{2.6}$$

where, in order to keep the four-dimensional Lorentz invariance, we allow a non-vanishing background value $\langle B_M^a \rangle (y^i)$ only for $M = i$, i.e. along the compact extra-dimensions. The presence of different background values along the Cartan subalgebra breaks the original

U(M) symmetry into $(\mathrm{U}(1))^M$. In terms of D branes this corresponds to generate M stacks, each consisting of one D brane with its own magnetization, different from that of the other branes, and the fields $\Phi_M^{ab}(x^\mu, y^i)$ for $M = i$ describe twisted open strings with the two end-points attached respectively to two D branes a and b having different magnetizations. If some of the background values are equal, then the original gauge group U(M) is broken into a product of non-abelian subgroups.

In the following we will not rewrite the entire action in terms of the fields introduced above, but we will only write the relevant terms, namely the quadratic terms involving the scalar and fermion fields and the trilinear terms involving a scalar and two fermions: we will derive the Kähler metrics from the former and the Yukawa couplings from the latter. We will also restrict our considerations to toroidal compactifications.

The quadratic terms for the fields $\Phi_M^{ab}(x^\mu, y^i)$ are the following:

$$S_2^{(\Phi)} = \frac{1}{2g^2}\int d^4x\sqrt{G_4}\int d^6y\sqrt{G_6}\Phi^{jba}\left[G^i{}_j\left(D_\mu D^\mu + \tilde{D}_k\tilde{D}^k\right) + 2i\langle(F_B)^i{}_j\rangle^{ab}\right]\Phi_i^{ab} \tag{2.7}$$

where

$$D_\mu\Phi_j^{ab} = \partial_\mu\Phi_j^{ab} - i(B_\mu^a - B_\mu^b)\Phi_j^{ab}\,; \qquad \tilde{D}_i\Phi_j^{ab} = \partial_i\Phi_j^{ab} - i\left(\langle B_i^a\rangle - \langle B_i^b\rangle\right)\Phi_j^{ab} \tag{2.8}$$

with

$$< (F_B)^i_j >^{ab} \equiv (F_B^a)^i{}_j - (F_B^b)^i{}_j \tag{2.9}$$

where $(F_B^a)^i{}_j$ is the field strength obtained from the background field B^a. Analogously we can consider the quadratic term for the fields $\delta B_i^a(x^\mu, y^i)$ obtaining:

$$S_2^{(\delta B)} = \frac{1}{2g^2}\int d^4x\sqrt{G_4}\int d^6y\sqrt{G_6}\delta B_i^a\left(\partial_j\partial^j + D_\mu D^\mu\right)\delta B^{ai} \tag{2.10}$$

where the gauge $\partial_M\delta B^{aM} = 0$ has been chosen.

The quadratic term of the fermions $\Psi^{ab}(x^\mu, y^i)$ is given by:

$$S_2^{(\Psi)} = \frac{i}{2g^2}\int d^4x\sqrt{G_4}\int d^6y\sqrt{G_6}\bar{\Psi}^{ba}\left(\Gamma^\mu D_\mu + \Gamma^i\tilde{D}_i\right)\Psi^{ab} \tag{2.11}$$

where $D_\mu\Psi^{ab}$ and $\tilde{D}_i\Psi^{ab}$ are the same as in eq. (2.8).

The trilinear Yukawa couplings are given by:

$$S_3^{(\Phi)} = \frac{1}{2g^2}\int d^4x\sqrt{G_4}\int d^6y\sqrt{G_6}\left(\bar{\Psi}^{ca}\Gamma^i\Phi_i^{ab}\Psi^{bc} - \bar{\Psi}^{ca}\Gamma^i\Phi_i^{bc}\Psi^{ab}\right) \tag{2.12}$$

and by

$$S_3^{(\delta B)} = \frac{1}{2g^2}\int d^4x\sqrt{G_4}\int d^6y\sqrt{G_6}\bar{\Psi}^{ab}(\not{\delta}B^b - \not{\delta}B^a)\Psi^{ba} \tag{2.13}$$

respectively for the twisted scalar Φ and for the untwisted scalar δB.

The four-dimensional effective action, corresponding to the ten-dimensional actions given above, is obtained by expanding the ten-dimensional fields as follows:[2]

$$\Phi_i^{ab}(X) = \sum_n \varphi_{n,i}^{ab}(x^\mu)\phi_n^{ab}(y^i)\,; \qquad \Psi^{ab}(X) = \sum_n \psi_n^{ab}(x^\mu) \otimes \eta_n^{ab}(y^i)\,. \tag{2.14}$$

The spectrum of the Kaluza-Klein states and their wave functions along the compact directions are obtained by solving the eigenvalue equations for the six-dimensional Laplace and Dirac operators:

$$-\tilde{D}_k \tilde{D}^k \phi_n^{ab} = m_n^2 \phi_n^{ab}\,, \qquad i\gamma^i_{(6)} \tilde{D}_i \eta_n^{ab} = \lambda_n\, \eta_n^{ab} \tag{2.15}$$

with the correct periodicity conditions along the compactified directions. It is worthwhile to remind that, according to a standard procedure followed in dimensionally reducing the ten-dimensional Dirac equation [11], it is necessary to make the operators $\Gamma^\mu D_\mu$ and $\Gamma^i \tilde{D}_i$ commute in order to properly define simultaneous eigenstates. This is accomplished by multiplying the latter operators with $\Gamma^{(5)} = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3$, which yields to eq. (2.15) after having used the decomposition:

$$\Gamma^\mu = \gamma^\mu_{(4)} \otimes \mathbb{I}_{(6)}\,, \qquad \Gamma^i = \gamma^5_{(4)} \otimes \gamma^i_{(6)}\,. \tag{2.16}$$

Inserting eq. (2.14) and the first equation in (2.15) in eq. (2.7) and using the coordinates z and $\bar{z}$ introduced in eq. (A.2) of appendix A for describing the torus T^2, one gets:[3]

$$\begin{aligned} S_2^{(\Phi)} &= \frac{1}{2g^2} \int d^4x \sqrt{G_4} \prod_{r=1}^{3} \left[(2\pi R)^2 \int d^2 z_r \sqrt{G^r} \right] \\ &\quad \times \sum_{r=1}^{3} \Phi_r^{j\;ba} \left[G^{r\;i}_{j} \left(D_\mu D^\mu - m_n^2 \right) + 2i \frac{< (F_r)_j^{\;i} >^{ab}}{(2\pi R)^2} \right] \Phi^{ab}_{ri} \end{aligned} \tag{2.17}$$

that, more explicitly, can be written as:

$$\begin{aligned} S_2^{(\Phi)} &= \frac{1}{2g^2} \int d^4x \sqrt{G_4} \sum_n \prod_{r=1}^{3} \left[(2\pi R)^2 \int d^2 z_r \sqrt{G^r} \right] \phi_n^{ba} \phi_n^{ab} \\ &\quad \times \left\{ \sum_{r=1}^{3} N_{\varphi_r}^2 \left[\varphi_{nr}^{ba,z}(x) \left[D_\mu D^\mu - m_n^2 + \frac{4\pi I_r}{(2\pi R)^2 \mathcal{T}_2^{(r)}} \right] \varphi_{nrz}^{ab}(x) \right] \right. \\ &\qquad \left. + \sum_{r=1}^{3} N_{\varphi_r}^2 \left[\varphi_{nr}^{ba,\bar{z}}(x) \left[D_\mu D^\mu - m_n^2 - \frac{4\pi I_r}{(2\pi R)^2 \mathcal{T}_2^{(r)}} \right] \varphi_{nr\bar{z}}^{ab}(x) \right] \right\} \end{aligned} \tag{2.18}$$

where we have used eqs. (B.5) and (B.6), in which the first Chern class I_r appears. Moreover, we have introduced a normalization factor N_{φ_r}, that in general will depend on the moduli. This factor has been fixed in ref. [8] requiring that the quadratic terms are canonically normalized. In this paper we adopt a different procedure and we will fix it later on by requiring the holomorphicity of the superpotential.

[2]The wave-function in the extra dimensions can in principle also depend on the index i, but this does not happen in our case as one can see from eqs. (2.15).

[3]In this formula the indices i and j run over each of the three torus T^2.

From eq. (2.18) we see that there are two towers of Kaluza-Klein states for each torus, with masses given by:

$$(M^{\pm}_{n,r})^2 = m_n^2 \pm \frac{4\pi I_r}{(2\pi R)^2 \mathcal{T}_2^{(r)}} = \frac{1}{(2\pi R)^2}\left[\sum_{s=1}^{3} \frac{2\pi |I_s|}{\mathcal{T}_2^{(s)}}(2N_s+1) \pm \frac{4\pi I_r}{\mathcal{T}_2^{(r)}}\right] \tag{2.19}$$

where N_s is an integer given by the oscillator number operator. The presence of the oscillator number is a consequence of the fact that, as shown in eq. (B.11), the Laplace operator can be written in terms of the creation and annihilation operators of an harmonic oscillator. Notice that, since we use a dimensionless $\mathcal{T}_2$, the factor $\frac{1}{(2\pi R)^2}$ in front is just there to cancel the dependence of the physical masses on the unphysical parameter R. One can have a massless state only if the following condition is satisfied for $I_r > 0$ or $I_r < 0$:[4]

$$\sum_{s=1}^{3} \frac{2\pi |I_s|}{\mathcal{T}_2^{(s)}} - \frac{4\pi |I_r|}{\mathcal{T}_2^{(r)}} = 0 \Longrightarrow \frac{1}{2}\sum_{s=1}^{3} \frac{|I_s|}{\mathcal{T}_2^{(s)}} - \frac{|I_r|}{\mathcal{T}_2^{(r)}} = 0\,. \tag{2.20}$$

In this case one keeps $\mathcal{N}=1$ supersymmetry because there is a massless scalar that is in the same chiral multiplet as a fermion that we will study later. If one of the I_r's is vanishing and the other two are equal, then we have an additional massless excitation corresponding to an extended $\mathcal{N}=2$ supersymmetry.

It is convenient to use fields φ^I with flat indices:[5]

$$\begin{aligned}
\varphi^{ab}_{\bar z} &= G_{\bar z z} e^{z}{}_I (\varphi^I)^{ab} \equiv \sqrt{\frac{\mathcal{T}_2}{2U_2}}\varphi^{ab}_+\,; \qquad (\varphi^{\bar z})^{ba} = e^{\bar z}{}_I (\varphi^I)^{ba} \equiv \sqrt{\frac{2U_2}{\mathcal{T}_2}}(\varphi^{ab}_+)^\dagger \\
\varphi^{ab}_{z} &= G_{z \bar z} e^{\bar z}{}_I (\varphi^I)^{ab} \equiv \sqrt{\frac{\mathcal{T}_2}{2U_2}}\varphi^{ab}_-\,; \qquad (\varphi^{z})^{ba} = e^{z}{}_I (\varphi^I)^{ba} \equiv \sqrt{\frac{2U_2}{\mathcal{T}_2}}(\varphi^{ab}_-)^\dagger
\end{aligned} \tag{2.21}$$

where

$$(\varphi_+)^{ab} = \left(\frac{\varphi^1 + i\,\varphi^2}{\sqrt{2}}\right)^{ab}; \qquad (\varphi_-)^{ab} = \left(\frac{\varphi^1 - i\,\varphi^2}{\sqrt{2}}\right)^{ab}; \qquad \varphi^{ba}_+ = (\varphi^{ab}_-)^\dagger\,. \tag{2.22}$$

The action for the twisted scalars restricted to the lowest modes of the two towers of Kaluza-Klein states becomes:

$$\begin{aligned}
S_2^{(\Phi_0)} = &-\frac{1}{2g^2}\prod_{s=1}^{3}\left[(2\pi R)^2 \int d^2 z_s \sqrt{G^{(z_s,\bar z_s)}}\right](\phi_0^{ab})^\dagger(\phi_0^{ab}) \int d^4x \sqrt{G_4}\sum_{r=1}^{3} N^2_{\varphi_r} \\
&\times \Big[(D_\mu(\varphi^{ab}_{r,+})^\dagger(x))(D^\mu \varphi^{ab}_{r,+}(x)) + (M^+_{0,r})^2 (\varphi^{ab}_{r,+})^\dagger(x)\varphi^{ab}_{r,+}(x) \\
&\quad + (D_\mu(\varphi^{ab}_{r,-})^\dagger(x))(D^\mu \varphi^{ab}_{r,-}(x)) + (M^-_{0,r})^2 (\varphi^{ab}_{r,-})^\dagger(x)\varphi^{ab}_{r,-}(x)\Big]\,.
\end{aligned} \tag{2.23}$$

The susy conditions given in eq. (2.20) show that only one of the two scalars is massless. In particular, by choosing in such equation $r=1$ and $I_1 > 0$, we see that $\varphi_{1,-}$ is the massless

[4]This condition is the field theory limit of the relation that one imposes in string theory in the twisted sector to keep $\mathcal{N}=1$ supersymmetry.

[5]We will discuss in the conclusions the reason of this choice.

scalar. The corresponding internal wave-function has been determined in ref. [8] and is the product of three eigenfunctions

$$\phi_0^{ab} = \prod_{r=1}^{3} \phi_{r,\text{sign}I_r}^{ab;n^r} \tag{2.24}$$

where

$$\begin{aligned} \phi_{r,+}^{ab;n_r} &= e^{\pi i I_r z_r \frac{\text{Im} z_r}{\text{Im} U^{(r)}}} \,\Theta \begin{bmatrix} \frac{2n_r}{I_r} \\ 0 \end{bmatrix} (I_r z_r | I_r U^{(r)}) \qquad \text{for } I_r > 0 \\ \phi_{r,-}^{ab;n_r} &= e^{i\pi |I_r| \bar{z}_r \frac{\text{Im} \bar{z}_r}{\text{Im} U^{(r)}}} \,\Theta \begin{bmatrix} \frac{-2n_r}{I_r} \\ 0 \end{bmatrix} (I_r \bar{z}_r | I_r \bar{U}^{(r)}) \qquad \text{for } I_r < 0 \end{aligned} \tag{2.25}$$

with $n_r = 0, \ldots, |I_r| - 1$ labelling the Landau levels. Instead, by taking $r = 1$ and $I_1 < 0$ we have that $\varphi_{1,+}$ becomes the massless mode. It is useful to notice that $(\phi_{r,+}^{ab;n_r})^\dagger = \phi_{r,-}^{ba;n_r}$, and furthermore, the reality of the scalar action implies:

$$\phi_0^{ba} = (\phi_0^{ab})^*. \tag{2.26}$$

In conclusion, by performing the Kaluza-Klein reduction of the low-energy world-volume action of a stack of D9 branes on $R^{3,1} \times T^2 \times T^2 \times T^2$, we have found two towers of Kaluza-Klein states for each of the scalar fields $\varphi_{r,\pm}$ for $r = 1, 2, 3$ corresponding to twisted or dycharged strings. In general, only the lowest state of one of the two towers and for a particular value of r (say $r = 1$ if eq. (2.20) is satisfied for $r = 1$) is massless, depending on the sign of I_1. We have now all the elements for computing the Kähler metric of the scalars $\varphi_\pm$. This will be done in section 3.

Next we consider eq. (2.10) for the adjoint scalars and expand the fluctuations as follows:

$$\delta B_i^a(x^\mu, y^i) = \sum_n C_{ni}^a(x^\mu) c_n^a(y^i)\,. \tag{2.27}$$

Inserting this expansion in eq. (2.10) and limiting ourselves to the constant zero mode we get:[6]

$$S_2^{(\delta B)} = \frac{1}{2g^2} \int d^4x \sqrt{G_4} \prod_{s=1}^{3} \left[(2\pi R)^2 \int d^2 z_s \sqrt{G^{(z_s, \bar{z}_s)}} \right] \sum_{r=1}^{3} C_r^{ai}(x) \left(\partial_\mu \partial^\mu + \partial_j \partial^j \right) C_{ri}^a(x)$$

that is equal to:

$$S_2^{(\delta B)} = -\frac{1}{2 \cdot 4\pi} \,\mathrm{e}^{-\phi_{10}} \prod_{s=1}^{3} T_2^{(s)} \int d^4x \sqrt{G_4} \left[\sum_{r=1}^{3} G_r^{ij} \partial^\mu C_{ri}^a(x) \partial_\mu C_{rj}^a(x) \right] \tag{2.28}$$

where we have taken the constant lowest eigenfunction $c^a(y^i) = 1$.

[6] Also here, as in eq. (2.17), with an abuse of notation, we take the indices i and j running over each of the three tori. Furthermore we let the subindex 0 drop.

Using the first metric in eq. (A.3) and going to the Einstein frame, we get the following final expression:

$$S_2^{(\delta B)} = -\frac{1}{2}\, \mathrm{e}^{-\phi_{10}}\, T_2^{(1)} T_2^{(2)} T_2^{(3)} \mathrm{e}^{2\phi_4} \int d^4x \sqrt{G_4} \left[\sum_{r=1}^{3} \frac{1}{T_2^{(r)} U_2^{(r)}} \partial^\mu \bar{\varphi}_r^a(x) \partial_\mu \varphi_r^a(x) \right] \tag{2.29}$$

where

$$\varphi_r^a \equiv i\frac{\bar{U}\tilde{C}_{r1}^a - \tilde{C}_{r2}^a}{\sqrt{4\pi}}\,. \tag{2.30}$$

Here, the fields $\tilde{C}_r$'s are the ones defined in the $\tilde{x}$ coordinate system introduced in appendix A.

We finally consider the kinetic term for the twisted fermions. It is obtained by plugging in eq. (2.11) the Kaluza-Klein mode expansion given in (2.14), getting:

$$S_F^{(2)} = \frac{1}{2g^2} \sum_{n,m} \int d^4x \sqrt{G^{(4)}} \left[\bar{\psi}_n^{ba} \left(i\gamma_{(4)}^\mu D_\mu + \lambda_n \gamma_{(4)}^5 \right) \psi_m^{ab} \right] \int d^6y \sqrt{G^{(6)}} (\eta_n^{ab})^\dagger\, \eta_m^{ab} \tag{2.31}$$

where we have used the identity $\bar{\eta}^{ba} = (\eta^{ab})^\dagger$ which follows from the structure of Γ^0 given in eq. (2.16).

3 The Kähler metrics

In this section we continue the calculation previously started for determining the Kähler metric of the scalars $\varphi_\pm$. In particular, we pay our attention on the term containing the massless scalar that we name φ. The Kähler metric Z can be read from the kinetic term for the field φ given by:

$$-\int d^4x \sqrt{G_4}\, Z(m, \bar{m})\, (D_\mu \bar{\varphi}(x))(D^\mu \varphi(x)) \tag{3.1}$$

written in the Einstein frame. The field φ is related to the fields φ_- by: $\varphi = \frac{\varphi_-}{\sqrt{4\pi}}$,[7] absorbing in the definition of the field the factor 4π present in g^2. m and $\bar{m}$ stand for the moduli.

By comparing this equation with eq. (2.23) the following expression for Z can be obtained:

$$Z(m, \bar{m}) = \frac{4\pi \mathrm{e}^{2\phi_4}}{2g^2} N_\varphi^2 \prod_{s=1}^{3} \left[(2\pi R)^2 \int d^2 z_s \sqrt{G^{(z_s, \bar{z}_s)}} \right] \phi_0^{ba} \phi_0^{ab} \tag{3.2}$$

where the factor $\mathrm{e}^{2\phi_4}$ has been added in order to go from the string to the Einstein frame.[8] N_φ is a normalization function that we have introduced in the previous section and that will be determined by requiring that the super-potential is holomorphic.

[7] We discuss only the case $I > 0$. The final relations are trivially extended to the case $I < 0$.

[8] The relation between the string and Einstein metric is $G_{\mu\nu}^{\text{string}} = \mathrm{e}^{2\phi_4} G_{\mu\nu}^{\text{Einstein}}$.

The integral over the six-dimensional compact space has been performed in ref. [8] with the following result valid both for positive and negative Chern-classes:

$$\prod_{r=1}^{3}\left[(2\pi R)^2\int d^2z_r\sqrt{G^r}\right]\phi_0^{ba}\phi_0^{ab} = \prod_{r=1}^{3}\left[\frac{(2\pi R)^2\mathcal{T}_2^{(r)}}{(2|I_r|U_2^{(r)})^{1/2}}\right]$$
$$= (2\pi\sqrt{\alpha'})^6\prod_{r=1}^{3}\left[\left(\frac{T_2^{(r)}}{2U_2^{(r)}}\right)^{1/2}\left(\frac{T_2^{(r)}}{|I_r|}\right)^{1/2}\right] \quad (3.3)$$

where, in going from the first to the second line, we have used eq. (A.7) and the last equation in (A.12) connecting the four-dimensional dilaton to the ten-dimensional one. Inserting eq. (3.3) in eq. (3.2) we get:

$$Z = \frac{e^{\phi_4}}{2}N_\varphi^2\prod_{r=1}^{3}\left[\left(\frac{1}{2U_2^{(r)}}\right)^{1/2}\left(\frac{T_2^{(r)}}{|I_r|}\right)^{1/2}\right] = \frac{N_\varphi^2}{2s_2^{1/4}}\prod_{r=1}^{3}\left[\frac{1}{(2u_2^{(r)})^{1/2}(t_2^{(r)})^{1/4}}\left(\frac{T_2^{(r)}}{|I_r|}\right)^{1/2}\right] \quad (3.4)$$

where the last equation in (A.13) has been used.

The scalars of the hypermultiplet can be obtained by imposing the following conditions:

$$\frac{|I_1|}{\mathcal{T}_2^{(1)}} = \frac{|I_2|}{\mathcal{T}_2^{(2)}}\,; \qquad I_3 = 0\,. \quad (3.5)$$

When they are satisfied, it is easy to see that we have two massless excitations corresponding to the two complex scalars of the hypermultiplet of $\mathcal{N}=2$ supersymmetry. One gets for them the following effective action:

$$-\frac{1}{2g^2}\int d^4x\sqrt{G_4}\left[N_{\varphi_1}^2(D_\mu\varphi_{1,-}^{ba}(x))(D^\mu\varphi_{1,+}^{ab}(x)) + N_{\varphi_2}^2(D_\mu\varphi_{2,-}^{ba}(x))(D^\mu\varphi_{2,+}^{ab}(x))\right]$$
$$\times\prod_{r=1}^{3}\left[(2\pi R)^2\int d^2z_r\sqrt{G^r}\right]\phi_0^{ba}\phi_0^{ab} \quad (3.6)$$

where now the wave function contains only the Θ-functions corresponding to the first two tori, while the wave function along the third torus is just a constant. From it, proceeding as above, we get:

$$\prod_{r=1}^{3}\left[(2\pi R)^2\int d^2z_r\sqrt{G^r}\right]\phi_0^{ba}\phi_0^{ab} = (2\pi\sqrt{\alpha'})^6T_2^{(3)}\prod_{r=1}^{2}\left[\left(\frac{T_2^{(r)}}{2U_2^{(r)}}\right)^{1/2}\left(\frac{T_2^{(r)}}{|I_r|}\right)^{1/2}\right]. \quad (3.7)$$

Introducing, as before, the two fields:

$$\varphi_1 = \frac{\varphi_{1,-}}{\sqrt{4\pi}}\,; \qquad \varphi_2 = \frac{\varphi_{2,-}}{\sqrt{4\pi}} \quad (3.8)$$

we can rewrite eq. (3.6) as follows:

$$-\int d^4x\sqrt{G_4}(m,\bar m)\left[Z_1^{\text{hyper}}(D_\mu\bar\varphi_1(x))(D^\mu\varphi_1(x)) + Z_2^{\text{hyper}}(D_\mu\bar\varphi_2(x))(D^\mu\varphi_2(x))\right] \quad (3.9)$$

where in the Einstein frame one has:

$$Z_i^{\rm hyper} = \frac{e^{2\phi_4}}{2} e^{-\phi_{10}} N_i^2 \prod_{r=1}^{3} T_2^{(r)} \prod_{r=1}^{2} \frac{1}{\left(2|I_r|\,U_2^{(r)}\right)^{1/2}}$$
$$= \frac{N_i^2}{2\left(4u_2^{(1)}u_2^{(2)}t_2^{(1)}t_2^{(2)}\right)^{1/2}} \prod_{r=1}^{2}\left(\frac{T_2^{(r)}}{|I_r|}\right)^{1/2} . \tag{3.10}$$

The normalization factors will be determined by imposing, as in the case of chiral matter, the holomorphicity of the superpotential.

We derive also the Kähler metric for the adjoint scalars. It can be obtained by comparing eqs. (3.1) and (2.29):

$$Z_r = e^{2\phi_4} e^{-\phi_{10}} \frac{T_2^{(1)}T_2^{(2)}T_2^{(3)}}{T_2^{(r)}U_2^{(r)}} = e^{\phi_4}\frac{(T_2^{(1)}T_2^{(2)}T_2^{(3)})^{1/2}}{T_2^{(r)}U_2^{(r)}} = \frac{e^{\phi_{10}}}{T_2^{(r)}U_2^{(r)}} = \frac{1}{t_2^{(r)}U_2^{(r)}} \tag{3.11}$$

and this expression agrees with eq. (2.20) of ref. [12] obtained from the DBI action.

In the final part of this section we compute the Kähler metric for twisted fermions which appears in the kinetic term for the fermions as:

$$\frac{i}{2}\int d^4x Z(m,\bar{m})\sqrt{G_4}\bar{\psi}^{ba}\,\gamma^{\mu}_{(4)} D_\mu\,\psi^{ab} . \tag{3.12}$$

Comparing it with eq. (2.31) we get:

$$Z = \frac{e^{2\phi_4}}{g^2} N_\psi^2 \int d^6y\sqrt{G_6}(\eta^{ab})^\dagger\,\eta^{ab} = \frac{e^{2\phi_4}e^{-\phi_{10}}}{4\pi} N_\psi^2 \prod_{r=1}^{3}\left[\left(\frac{T_2^{(r)}}{|I_r|}\right)^{1/2}\left(\frac{T_2^{(r)}}{2U_2^{(r)}}\right)^{1/2}\right]$$
$$= \frac{e^{\phi_4}}{4\pi} N_\psi^2 \prod_{r=1}^{3}\left[\left(\frac{T_2^{(r)}}{|I_r|}\right)^{1/2}\left(\frac{1}{2U_2^{(r)}}\right)^{1/2}\right] \tag{3.13}$$

where the factor $e^{2\phi_4}$ has been added for going to the Einstein frame. Eq. (3.13) gives the same dependence on the moduli as the eq. (3.4) does, with the only difference due to a constant normalization factor. This is an expected result which follows from $\mathcal{N}=1$ supersymmetry, since the fields φ and ψ belong to the same chiral multiplet. We also deduce that $N_\varphi = N_\psi/\sqrt{2\pi}$.

In conclusion, with our procedure we have determined how the Kähler metrics explicitly depend on the moduli apart from that normalization factor that we will determine in the next section by requiring the holomorphicity of the superpotential.

4 Yukawa couplings

In this section we evaluate the Yukawa couplings both for the chiral multiplet and the hypermultiplet. In the case of the chiral multiplet, we start from the action in eq. (2.12)

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where the expansions in eq. (2.14) have been inserted:

$$
\begin{aligned}
S_3^{(\Phi)} &= \frac{1}{2g^2}\int d^4x\sqrt{G_4}\int d^6y\sqrt{G_6}\sum_{n,m,l}\bar{\psi}_n^{ca}\,\gamma_{(4)}^5 \\
&\quad\times\left[\varphi_{i,m}^{ab}\,\psi_l^{bc}\otimes(\eta_n^{ac})^\dagger\gamma_{(6)}^i\phi_m^{ab}\eta_l^{bc}-\varphi_{i,m}^{bc}\,\psi_l^{ab}\otimes(\eta_n^{ac})^\dagger\gamma_{(6)}^i\phi_m^{bc}\eta_l^{ab}\right]. \qquad (4.1)
\end{aligned}
$$

In the following we focus on the term containing the massless scalar relative to the first torus. This implies that the condition:

$$
\frac{|I_1^{ab}|}{T_2^{(1)}}=\frac{|I_2^{ab}|}{T_2^{(2)}}+\frac{|I_3^{ab}|}{T_2^{(3)}} \qquad (4.2)
$$

must be satisfied. We are allowed to choose I_1^{ab} being positive and consequently the massless scalar results to be $\varphi_1=\frac{\varphi_{1,-}}{\sqrt{4\pi}}$. Furthermore, in order to satisfy the condition

$$
I_r^{ab}+I_r^{bc}+I_r^{ca}=0 \qquad (4.3)
$$

and to have non-zero Yukawa couplings, on the first torus we fix

$$
I_1^{ca}<0\,; \qquad I_1^{ab}>0\,; \qquad I_1^{bc}<0 \qquad (4.4)
$$

implying that the internal wave function associated with the bosonic zero mode solution is the first one in eq. (2.25).

In this case, as it is shown in appendix C, one is left with the following expression:

$$
S_3^{(\Phi)}=\int d^4x\sqrt{G_4}\bar{\psi}^{ca}\,\gamma_{(4)}^5\varphi_1^{ab}\,\psi^{bc}Y^s \qquad (4.5)
$$

with the Yukawa coupling in the string frame given by:

$$
\begin{aligned}
Y^s &= \frac{N_{\varphi_1}^{ab}N_\psi^{ca}N_\psi^{bc}}{\sqrt{2}g^2}\sqrt{4\pi}\prod_{r=1}^{3}\left[(2\pi R)^2\int d^2z_r\sqrt{G^r}\right] \\
&\quad\times(\eta_{1,-}^{ca}\,\phi_{1,+}^{ab}\,\eta_{1,-}^{bc})(\pm\eta_{2,\mp}^{ca}\,\phi_{2,\mathrm{sign}\,I_2^{ab}}^{ab}\,\eta_{2,\pm}^{bc})(\pm\eta_{3,\mp}^{ca}\,\phi_{3,\mathrm{sign}\,I_3^{ab}}^{ab}\,\eta_{3,\pm}^{bc}) \qquad (4.6)
\end{aligned}
$$

where in each of the last two tori we can choose the upper and lower sign independently from each other. The important point is that Y^s results to be non vanishing only when the chiralities of the two spinors $\eta_{2,3}^{ca}$ and $\eta_{2,3}^{bc}$ are opposite, while those of η_1^{ca} and η_1^{bc} are equal. As in the case of the Kähler metric we add the three normalization factors for each of the three four-dimensional fields that we will determine by requiring the holomorphicity of the superpotential.

The integral on T^2 has been computed in appendix C. It can be generalized, by using eqs. (C.22)–(C.27), to the case of $T^2\times T^2\times T^2$ for arbitrary values of the Chern classes as follows:

$$
\begin{aligned}
Y^s = \frac{\mathrm{e}^{-\phi_{10}}}{\sqrt{8\pi}}\sigma N_\varphi N_\psi N_\psi\prod_{r=1}^{3}\Bigg\{&\frac{T_2^{(r)}}{\left(2U_2^{(r)}|I_r^{ab}|\chi_r^{ab}|I_r^{bc}|\chi_r^{bc}|I_r^{ca}|\chi_r^{ca}\right)^{1/2}} \\
&\times\Theta\begin{bmatrix}2\left(\frac{n'}{I_r^{ca}}+\frac{m'}{I_r^{bc}}+\frac{l'}{I_r^{ab}}\quad\right)\\ 0\end{bmatrix}(0|-I_r^{ab}I_r^{bc}I_r^{ca}U_f^{(r)})\Bigg\} \qquad (4.7)
\end{aligned}
$$

with $n' = 0, \ldots, |I^{ca}| - 1$; $m' = 0, \ldots, |I^{bc}| - 1$; $l' = 0, \ldots, |I^{ab}| - 1$. Moreover χ_r is defined in eq. (C.26) and

$$U_f^{(r)} = \begin{cases} U^{(r)} & \text{for} \quad \text{sign}(I^{ca} I^{bc} I^{ab}) < 0 \\ \bar{U}^{(r)} & \text{for} \quad \text{sign}(I^{ca} I^{bc} I^{ab}) > 0 \,. \end{cases} \tag{4.8}$$

The previous result agrees with the one in ref. [8] and each of the three Θ-functions in eq. (4.7) is a holomorphic function of the complex structure of the corresponding torus. It is worth noticing that, because of eq. (4.8), if $\text{sign}(I_r^{ca} I_r^{bc} I_r^{ab})$ is not the same for each value of $r = 1, 2, 3$, then some of Θ-functions will depend on U, while others will depend on $\bar{U}$.

Notice that eq. (4.7) is valid not only for the choice in eq. (4.4), but for any arbitrary choice of the Chern classes. One can rewrite the Yukawa couplings in the Einstein frame multiplying the equation by $\mathrm{e}^{4\phi_4}\mathrm{e}^{-\phi_4}$, where the first factor comes from the rescaling of the square root of the determinant of the metric and the second from that of the two fermionic fields. Using the last equation in (A.12) and taking then into account eq. (A.15), we see that the factors containing ϕ_4 combine together with other factors to give:

$$\begin{aligned} Y^E = \frac{\mathrm{e}^{K/2}}{\sqrt{8\pi}} \sigma N_\varphi N_\psi N_\psi \prod_{r=1}^{3} \Bigg\{ & \frac{(T_2^{(r)})^{1/2}}{\left(2|I_r^{ab}|\chi_r^{ab}|I_r^{bc}|\chi_r^{bc}|I_r^{ca}|\chi_r^{ca}\right)^{1/2}} \\ & \times \Theta \begin{bmatrix} 2\left(\frac{n'}{I_r^{ca}} + \frac{m'}{I_r^{bc}} + \frac{l'}{I_r^{ab}}\right) \\ 0 \end{bmatrix} (0| - I_r^{ab} I_r^{bc} I_r^{ca} U_f^{(r)}) \Bigg\} \end{aligned} \tag{4.9}$$

where K is the Kähler potential given in eq. (A.14). With the choice in eqs. (4.2) and (4.4) the previous equation becomes:

$$\begin{aligned} Y^E = & \frac{\mathrm{e}^{K/2}}{\sqrt{8\pi}} \sigma N_{\varphi_1}^{ab} N_\psi^{ca} N_\psi^{bc} \frac{(T_2^{(1)})^{1/2}}{(2I_1^{ab})^{1/2}} \prod_{r=2}^{3} \frac{(T_2^{(r)})^{1/2}}{\left(2|I_r^{bc}|\chi_r^{bc}|I_r^{ca}|\chi_r^{ca}\right)^{1/2}} \\ & \times \prod_{r=1}^{3} \Theta \begin{bmatrix} 2\left(\frac{n'}{I_r^{ca}} + \frac{m'}{I_r^{bc}} + \frac{l'}{I_r^{ab}}\right) \\ 0 \end{bmatrix} (0| - I_r^{ab} I_r^{bc} I_r^{ca} U_f^{(r)}) \,. \end{aligned} \tag{4.10}$$

Because of the terms depending on the magnetizations, eq. (4.10) is not a holomorphic function of the moduli unless we choose the normalization factors $N_{\varphi_1}^{ab}, N_\psi^{bc}, N_\psi^{ca}$ in such a way to eliminate such dependence. This is what we are going to explain in the following.

Eq. (4.3) must be satisfied for the three tori. In the first torus we have chosen the I's as in eq. (4.4). In the second torus let us choose $\text{sign}(I_2^{ab}) = \text{sign}(I_2^{bc})$.[9] With these two choices we get:

$$I_1^{ab} + I_1^{bc} + I_1^{ca} = 0 \Longrightarrow |I_1^{bc}| + |I_1^{ca}| = |I_1^{ab}| \Longrightarrow \nu_1^{ab} = \nu_1^{bc} + \nu_1^{ca} \tag{4.11}$$

$$I_2^{ab} + I_2^{bc} + I_2^{ca} = 0 \Longrightarrow |I_2^{ab}| + |I_2^{bc}| = |I_2^{ca}| \Longrightarrow \nu_2^{ca} = \nu_2^{ab} + \nu_2^{bc} \tag{4.12}$$

because $\text{sign}(I_2^{ca}) = -\,\text{sign}(I_2^{bc})$, as follows from eq. (4.6). In the last step we have used the definition:

$$\pi \nu_r \equiv \frac{|I_r|}{T_2^{(r)}} \tag{4.13}$$

[9] If this is not the case, we can repeat what we are going to do, substituting I_2^{bc} with I_2^{ca} without loss of generality.

where the quantities ν_r will be shown to have also a precise meaning in the theory of magnetized branes and strings generating our model at low-energy. Finally, on the third torus let us choose $\mathrm{sign}(I_3^{ab}) = -\mathrm{sign}(I_3^{bc})$. This means that:

$$I_3^{ab} + I_3^{bc} + I_3^{ca} = 0 \Longrightarrow |I_3^{ab}| + |I_3^{ca}| = |I_3^{bc}| \Longrightarrow \nu_3^{bc} = \nu_3^{ab} + \nu_3^{ca} \tag{4.14}$$

because $\mathrm{sign}(I_3^{ca}) = -\,\mathrm{sign}(I_3^{bc})$, as follows from eq. (4.6).[10]

Summing eqs. (4.11), (4.12) and (4.14) we get:

$$(\nu_1^{ab} - \nu_2^{ab} - \nu_3^{ab}) - (\nu_1^{bc} + \nu_2^{bc} - \nu_3^{bc}) + (-\nu_1^{ca} + \nu_2^{ca} - \nu_3^{ca}) = 0 \tag{4.15}$$

which is satisfied by taking:

$$\nu_1^{ab} = \nu_2^{ab} + \nu_3^{ab}\,; \qquad \nu_3^{bc} = \nu_1^{bc} + \nu_2^{bc}\,; \qquad \nu_2^{ca} = \nu_1^{ca} + \nu_3^{ca}\,. \tag{4.16}$$

Such a configuration preserves $\mathcal{N} = 1$ supersymmetry in all the three sectors ab, bc and ca. With the previous choices one gets:

$$\chi_2^{bc} = \chi_3^{ca} = 0\,; \qquad \chi_3^{bc} = \chi_2^{ca} = 1\,. \tag{4.17}$$

Using these values in eq. (4.10) we see that, if the normalization factors are taken as follows:

$$N_{\varphi_1}^{ab} = \left(\frac{|I_1^{ab}|}{T_2^{(1)}}\right)^{1/2} \hat{N}_{\varphi_1}^{ab}\,; \qquad N_{\psi}^{ca} = \left(\frac{|I_2^{ca}|}{T_2^{(2)}}\right)^{1/2} \hat{N}_{\psi}^{ca}\,; \qquad N_{\psi}^{bc} = \left(\frac{|I_3^{bc}|}{T_2^{(3)}}\right)^{1/2} \hat{N}_{\psi}^{bc} \tag{4.18}$$

with the product $\hat{N}_{\varphi_1}^{ab}\hat{N}_{\psi}^{ca}\hat{N}_{\psi}^{bc}$ being independent on the moduli, then the Yukawa coupling becomes a holomorphic function of the moduli! This means that the three factors $\hat{N}$ can in general depend on the magnetizations $\frac{|I_r|}{T_2^{(r)}}$ in such a way, however, that this dependence is cancelled when we take their product. But, in order that this could happen, it is required that the normalization of the scalar and of its fermionic partner corresponding to the torus T_r^2 be not only a function of the magnetization on T_r^2, but also on the magnetization on the other two tori. This seems to us unlikely, but cannot be excluded in principle. In the following we assume that this does not happen, but it is clear that the Kähler metrics that we will derive, depends on this assumption.

In the final part of this section we consider the Yukawa coupling involving the two fermions of the hypermultiplet and a scalar living in the adjoint representation of the gauge group.[11] This coupling is obtained by compactifying the terms of ten dimensional action given in eq. (2.13). In the following we restrict our analysis only to the first term of this equation which gives the interaction of the two fermions $\bar{\psi}_\alpha^{ab}$ and ψ_β^{ba} living in the bifundamental representation of the gauge group $U_a(1) \times U_b(1)$ with the massless scalar in the "adjoint" representation of the second gauge group. Here, the indices α and β label

[10] For any other choice of signs we get either an equivalent realization of $\mathcal{N} = 1$ supersymmetry or an extended $\mathcal{N} = 4$ supersymmetry that we do not consider here.

[11] We call it adjoint with an abuse of notation having in mind that the U(1) gauge group is extended to a non-abelian group when some of the background values are equal to each other as discussed in section 2.

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the degeneracy of the lowest fermionic state as described at the end of appendix B. The second term of eq. (2.13) gives the interaction of the two fermions with the scalar in the "adjoint" of the group $U_a(1)$. Such interaction term has a sign that is opposite to the first term. This sign can be taken into account by multiplying the Yukawa coupling by a factor σ which is equal to $+1$ $[-1]$ if the fermion ψ^{ba} is in the fundamental representation of the gauge group $U_b(1)$ $[U_a(1)]$.

Inserting in the first term of eq. (2.13) the zero mode of the expansion in eq. (2.27) and the massless fermions given in eq. (B.39) we get:

$$\mathcal{S}^{\delta B}_{3;\,\alpha,\,\beta} = \int d^4x\sqrt{G_4}\bar{\psi}^{ab}_\alpha \sum_{r=1}^{3}[\varphi^a_r\;(Y^r_{\alpha,\,\beta})^s + \bar{\varphi}^a_r\;(\bar{Y}^r_{\alpha,\,\beta})^s]\;\gamma^5_{(4)}\psi^{ba}_\beta \tag{4.19}$$

where we have taken the internal wave function of the scalar to be equal to 1 and used the relation between the four-dimensional fields and ten-dimensional ones given in eq. (A.9). The Yukawa coupling in the four-dimensional string frame is equal to:

$$(Y^r_{\alpha,\,\beta})^s = \sqrt{4\pi}\,\sigma\frac{\sqrt{\alpha'}}{R}\,\frac{N_{\psi_\alpha}N_{\psi_\beta}}{2g^2}\prod_{r=1}^{3}\left[(2\pi R)^2\int d^2z^r\sqrt{G^{(z^r,\,\bar{z}^r)}}\right](\eta^{ba}_\alpha)^\dagger\left[\frac{1}{2U_2^{(r)}}\gamma^{z^r}_{(6)}\right]\eta^{ba}_\beta \tag{4.20}$$

with $(\bar{Y}^r_{\alpha,\,\beta})^s = ((Y^r_{\beta,\,\alpha})^\dagger)^s$. The factors depending on R and $\sqrt{\alpha'}$ are a direct consequence of the factors present in eq. (A.9).

Due to the peculiar structure of the six-dimensional Γ matrices, the only terms of the previous expression, that are different from zero, are the ones with $r = 3$ and $\alpha \neq \beta$. The result is:

$$(Y^3_{\uparrow,\downarrow})^s = (Y^3_{\downarrow,\uparrow})^s = \sigma\,N_{\psi_\downarrow}N_{\psi_\uparrow}\frac{e^{-\phi_{10}}}{2\sqrt{4\pi}}\prod_{r=1}^{3}T_2^{(r)}\prod_{r=1}^{2}\left[\frac{1}{(2U_2^{(r)}|I^{ab}_r|)^{1/2}}\right]\sqrt{\frac{1}{U_2^{(3)}T_2^{(3)}}} \tag{4.21}$$

where we have used the relation between T_2 and $\mathcal{T}_2$ given in eq. (A.7) together with the expression of the Dirac-matrix given in eq. (B.27).

The Yukawa coupling in the Einstein frame is obtained by multiplying the previous expression by the dilaton factor $e^{3\phi_4}$. Introducing, as for the case $\mathcal{N} = 1$, the Kähler potential K given in eq. (A.15) we get:

$$(Y^3_{\uparrow,\downarrow})^E = (Y^3_{\downarrow,\uparrow})^E = \sigma\,N_{\psi_\downarrow}N_{\psi_\uparrow}\frac{e^{K/2}}{4\sqrt{4\pi}}\prod_{r=1}^{2}\left(\frac{T_2^{(r)}}{|I^{ab}_r|}\right)^{1/2}. \tag{4.22}$$

The previous coupling is not a holomorphic function of the moduli. However, normalizing the fermions as follows:

$$N_{\psi_\uparrow} = N_{\psi_\downarrow} = \left(\frac{|I^{ab}_1|}{T_2^{(1)}}\right)^{1/2} = \left(\frac{|I^{ab}_2|}{T_2^{(2)}}\right)^{1/2} \tag{4.23}$$

where the first relation in eq. (3.5) has been explicitly used, we restore also in this case the holomorphicity of the super-potential.

5 Fixing the Kähler metrics

In the previous section we have fixed the normalization of the four-dimensional fields by requiring that the Yukawa couplings come from a holomorphic superpotential. We can now go back to the Kähler metrics that we have determined in section 3 apart from an overall normalization, and fix them uniquely using the value obtained from the Yukawa couplings. Let us start from the chiral multiplet where the Kähler metric is given in eq. (3.4). Inserting in it the normalization given in the first equation in (4.18) we get:

$$Z_{ab}^{\rm chiral} = \frac{1}{2 s_2^{1/4}} \prod_{r=1}^{3} \left[\frac{1}{(2u_2^{(r)})^{1/2} (t_2^{(r)})^{1/4}} \right] \left(\frac{\nu_1^{ab}}{\pi \nu_2^{ab} \nu_3^{ab}} \right)^{1/2} \tag{5.1}$$

where eq. (4.13) has been used. Let us now examine the dependence on the magnetizations, given by the last factor in the r.h.s. of the previous equation. The dependence of the Kähler metric on the magnetizations has been computed by means of a pure string calculation in refs. [3, 4] obtaining in our notations:

$$\left[\frac{\Gamma(1-\nu_1^{ab})}{\Gamma(\nu_1^{ab})} \frac{\Gamma(\nu_2^{ab})}{\Gamma(1-\nu_2^{ab})} \frac{\Gamma(\nu_3^{ab})}{\Gamma(1-\nu_3^{ab})} \right]^{1/2} \Longrightarrow \left(\frac{\nu_1^{ab}}{\nu_2^{ab} \nu_3^{ab}} \right)^{1/2} . \tag{5.2}$$

This expression, in the limit of small magnetizations, coincides with our result in eq. (5.1) for the part concerning the magnetizations, consistently with the fact that this is just the limit that one should perform in going from the string to the field theory definition of ν. In this limit, for positive values of ν, one has:

$$\tan \pi \nu_r = \frac{|I_r|}{T_2^{(r)}} \Longrightarrow \pi \nu_r = \frac{|I_r|}{T_2^{(r)}} \tag{5.3}$$

which is realized for small values of ν_r. The expression for the twisted Kähler metric obtained from considerations about holomorphicity within the instanton calculus [5–7] contained a possible additional dependence on the magnetizations that we do not find in our field theoretical procedure.

Turning to the Kähler metric of the hypermultiplet, given in eq. (3.10), we see that the dependence on the magnetization cancels and we get:

$$Z_i^{\rm hyper} = \frac{1}{2\left(4 u_2^{(1)} u_2^{(2)} t_2^{(1)} t_2^{(2)}\right)^{1/2}} . \tag{5.4}$$

In this case the dependence on the magnetization drops out in agreement with the well-known result (see for instance eq. (2.45) of ref. [12]).

6 Conclusions and outlook

In this paper we have proposed a procedure for determining the Kähler metric for the twisted open strings defined as the ones having their end-points attached to two D branes with different magnetizations. Unlike ref. [8], where the kinetic terms are canonically normalized and then the Kähler metrics appear in the Yukawa couplings, we keep for the quadratic terms the normalization that comes naturally from the Kaluza-Klein reduction apart from a normalization factor that we then determine requiring that the Yukawa couplings correspond to a holomorphic superpotential. We find that these normalization factors depend only on the magnetization. This procedure yields the Kähler metrics proposed in the literature [5–7] without the arbitrary factors that appeared in the previously mentioned proposals. In particular, our procedure allows us to correctly determine the Kähler metric for the hypermultiplet that agrees with the expression obtained with other methods.

In deriving the previous results we have, however, made implicitly two assumptions. The first one is that the normalization factor contains only the minimal number of factors that make the superpotential holomorphic and the second one is that our reasonings are based on the specific form of the scalar fields $\varphi_{r,\pm}$ that we use (see eqs. (2.21) and (2.22)). But why do we use these scalar fields? Before trying to answer this question, let us observe that the introduction of the normalization factor allows us to actually rescale the field with a quantity and at the same time rescale the normalization factor with the inverse quantity without changing the Kähler metrics and the Yukawa couplings. In particular, this rescaling factor can be a function of the moduli. This means that the presence of the normalization factor does not allow us to determine the absolute normalization of the scalar field.

Having said this, let us find the relation of $\varphi_{r,-}$ with the original ten-dimensional fields. In the case of the adjoint scalars such relation is given in eq. (2.30). For the twisted fields, starting from eq. (2.21) and then using the relation between the variables x^1, x^2 and $z, \bar{z}$ given in appendix A, we get:

$$\begin{aligned}\varphi_{r\,-} &= \sqrt{\frac{2\,U_2^{(r)}}{\mathcal{T}_2^{(r)}}}\varphi_{r\,z} = \sqrt{\frac{2\,U_2^{(r)}}{\mathcal{T}_2^{(r)}}}\left(\frac{\partial x^{2r+2}}{2\pi R\,\partial z^r}W_{2r+2} + \frac{\partial x^{2r+3}}{2\pi R\,\partial z^r}W_{2r+3}\right)\\ &= \frac{i}{\sqrt{2U_2^{(r)}\mathcal{T}_2^{(r)}}}(\bar{U}^{(r)}W_{2r+2} - W_{2r+3}) \end{aligned} \tag{6.1}$$

where we have used eqs. (A.2) and the transformation rule of a covariant vector:[12]

$$W_{z^r} = \frac{\partial x^k}{2\pi R\partial z^r}W_k\,. \tag{6.2}$$

Unlike the adjoint scalar in eq. (2.30), the fundamental scalar in eq. (6.1) is not a holomorphic function of the fields for the presence of the non-holomorphic pre-factor. If we want a holomorphic function we can incorporate the extra non-holomorphic factor in the normalization factor fixing it in a unique way. This requirement eliminates the possibility of rescaling both the scalar field and the normalization factor, as discussed above. This unique rescaling leaves both the Kähler metric and the Yukawa couplings, determined above, unchanged.

[12] The extra factor $2\pi R$ in eq. (6.1) is necessary for dimensional reasons (the x variables are dimensional, while the z variables are dimensionless).

The procedure outlined in this paper can be extended to more complicated and more realistic compact manifolds and, if we restrict ourselves to toroidal compactifications, it would be important to develop string techniques for fully reproducing the field theoretical results in the zero slope limit and also for computing string corrections to the field theory behavior.

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A The torus T^2

In this appendix we summarize the properties of the torus T^2 and we list the combination of the string moduli that enter in supergravity.

The torus T^2 can be equivalently described either by the "curved" dimensional coordinates x^1, x^2 that are periodic with period $2\pi R$ going around the two one-cycles of the torus

$$x^1 \equiv x^1 + 2\pi R \qquad x^2 \equiv x^2 + 2\pi R \tag{A.1}$$

or by the "flat" dimensionless coordinates $z, \bar{z}$ given by:

$$z = \frac{x^1 + Ux^2}{2\pi R} \qquad \bar{z} = \frac{x^1 + \bar{U}x^2}{2\pi R}\,. \tag{A.2}$$

The dimensional parameter R is arbitrary and has been introduced to deal with dimensionless z and $\bar{z}$. We will see, however, that the physical quantities do not depend on R. The metric of the torus in the two coordinate systems is equal to:

$$G_{ij}^{(x^1,x^2)} = \frac{\mathcal{T}_2}{U_2}\begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}; \qquad G_{ij}^{(z,\bar{z})} = \frac{\mathcal{T}_2}{2U_2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.3}$$

They imply

$$ds^2 = G_{ij}^{(x^1,x^2)} dx^i dx^j = \frac{\mathcal{T}_2}{U_2}|dx^1 + U dx^2|^2 = (2\pi R)^2 \frac{\mathcal{T}_2}{U_2} dz d\bar{z}\,. \tag{A.4}$$

The complex quantities $U = U_1 + iU_2$ and $\mathcal{T} = \mathcal{T}_1 + i\mathcal{T}_2$ correspond, respectively, to the complex and the Kähler structures of the torus T^2. The real part of the Kähler structure $\mathcal{T}_1$ is related to the the Kalb-Ramond field by $\mathcal{T}_1 = -B_{12}$, while its imaginary part is related to the volume of the torus. We use dimensionless moduli. They are given in terms of the physical parameters of the torus, consisting of two radii R_1 and R_2 and an angle α, by the following expressions:

$$U = \frac{R_2}{R_1}\mathrm{e}^{i\alpha}; \qquad \mathcal{T}_2 = \frac{R_1 R_2}{R^2}\sin\alpha\,. \tag{A.5}$$

The area of the torus T^2 is given by:

$$\begin{aligned}\mathcal{A} &= \int_0^{2\pi R} dx^1 \int_0^{2\pi R} dx^2 \sqrt{G^{(x^1,x^2)}} = (2\pi)^2 R_1 R_2 \sin\alpha \\ &= (2\pi R)^2 \int_{T^2} d^2z \sqrt{G^{(z,\bar{z})}} = (2\pi R)^2 \mathcal{T}_2 \end{aligned} \tag{A.6}$$

and is independent of R.

In string theory one usually introduces "curved" dimensional coordinates $\tilde{x}^1$, $\tilde{x}^2$ which have periodicity $2\pi\sqrt{\alpha'}$ when translated along the two one-cycles of the torus.[13] The relation between these coordinates and the ones given in eq. (A.1) is $\tilde{x} = (\sqrt{\alpha'}/R)x$. In terms of these coordinates the volume of the torus is measured in units of the string length $2\pi\sqrt{\alpha'}$ rather then $2\pi R$. This means that the string Kähler structure T_2 is given by eq. (A.5) with R substituted by $\sqrt{\alpha'}$, i.e.

$$T_2 = \frac{R^2}{\alpha'} \mathcal{T}_2 \,. \tag{A.7}$$

The relation between the covariant fields defined in the x-coordinates with the corresponding ones in the $\tilde{x}$-coordinates can be obtained from the general transformation of coordinates rule of covariant fields:

$$\tilde{C}_r = \left(\frac{\partial x^s}{\partial \tilde{x}^r}\right) C_s = \frac{R}{\sqrt{\alpha'}} C_r \,. \tag{A.8}$$

It is also useful to give the relation between the four-dimensional field φ defined in eq. (2.30) and the scalars written in the complex system of coordinates C_z. From its definition, given in eq. (2.30), we can write:

$$\varphi = i\frac{R\bar{U}}{\sqrt{4\pi\alpha'}} \left(\frac{2\pi R \partial z}{\partial x^1}\right) C_z - i\frac{R}{\sqrt{4\pi\alpha'}} \left(\frac{2\pi R \partial z}{\partial x^2}\right) C_z = \frac{2U_2 R}{\sqrt{4\pi\alpha'}} C_z \tag{A.9}$$

where the extra factor $2\pi\, R$ is necessary for dimensional reasons, because the z's, differently from the x's, are dimensionless coordinates.

We can introduce the following vierbein and its inverse:

$$e^I{}_i = \frac{1}{2}\sqrt{\frac{\mathcal{T}_2}{U_2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}; \qquad i = z,\, \bar{z}\,; \qquad e^i{}_I = \sqrt{\frac{U_2}{\mathcal{T}_2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \tag{A.10}$$

such that

$$G^{(z,\bar{z})}_{ij} = e^J{}_i \delta_{JI} e^I{}_j \equiv (e^t\, e)_{ij} = \frac{\mathcal{T}_2}{2U_2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.11}$$

In the final part of this appendix we introduce the moduli fields that one should use in the supergravity action. In string theory the moduli are the ten dimensional dilaton and

[13] For the sake of simplicity we could have introduced torus coordinates with the same periodicity $R = \sqrt{\alpha'}$. It is, however, useful to keep them different from each other to have a check on the formulas because the physics is independent on their choice.

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the ones related to the complex and Kähler structure $U = U_1 + iU_2$ and $T = T_1 + iT_2$. In supergravity the variables to use are instead the following:

$$s_2 = \mathrm{e}^{-\phi_{10}} \prod_{r=1}^{3} T_2^{(r)}; \quad t_2^{(r)} = \mathrm{e}^{-\phi_{10}} T_2^{(r)}; \quad u_2 = U_2\,; \quad \mathrm{e}^{-\phi_4} = \mathrm{e}^{-\phi_{10}} \prod_{r=1}^{3} \left(T_2^{(r)}\right)^{1/2}. \tag{A.12}$$

The subindex 2 means that they are the imaginary part of a complex quantity whose real part is given by suitable RR fields that are not needed here. The previous relations imply the following:

$$\frac{s_2}{\prod_{r=1}^3 t_2^{(r)}} = \mathrm{e}^{2\phi_{10}}; \quad \prod_{r=1}^{3} T_2^{(r)} = \frac{s_2^{3/2}}{\left(\prod_{r=1}^3 t_2^{(r)}\right)^{1/2}}; \quad \mathrm{e}^{2\phi_4} = s_2^{-1/2} \left(\prod_{i=1}^{3} t_2^{(i)}\right)^{-1/2}. \tag{A.13}$$

The Kähler potential of the closed string moduli is given by:

$$K = -\log s_2 - \sum_{r=1}^{3} \left[\log t_2^{(r)} + \log u_2^{(r)}\right]. \tag{A.14}$$

It satisfies the following identity:

$$e^{K/2} = \frac{e^{2\phi_4}}{\prod_{r=1}^3 (U_2^r)^{1/2}}. \tag{A.15}$$

B Solving the eigenvalue equations

Let us start analyzing the case of the torus T^2. In terms of the variables $z, \bar{z}$ defined in the appendix A the gauge covariant derivative is given by:

$$\tilde{D}_z = \partial_z - iB_z\,; \qquad \tilde{D}_{\bar{z}} = \partial_{\bar{z}} - iB_{\bar{z}} \tag{B.1}$$

where the background fields B_z and $B_{\bar{z}}$ are given by:

$$B_z = \frac{\pi I \bar{z}}{(U - \bar{U})}; \qquad B_{\bar{z}} = -\frac{\pi I z}{(U - \bar{U})} \Longrightarrow B = B_z dz + B_{\bar{z}} d\bar{z} = \frac{\pi I(\bar{z}dz - zd\bar{z})}{2iU_2}. \tag{B.2}$$

They imply ($F \equiv dB$):

$$\left[-i\tilde{D}_z, -i\tilde{D}_{\bar{z}}\right] = -\frac{\pi I}{U_2} \equiv iF_{z\bar{z}}\,. \tag{B.3}$$

The expression for $F_{z\bar{z}}$ can be obtained from the fact that the first Chern class must be an integer I:

$$\int \frac{F}{2\pi} = \int F_{z\bar{z}} dz \wedge d\bar{z} = I \Longrightarrow F_{z\bar{z}} = -\frac{\pi I}{iU_2}. \tag{B.4}$$

From the previous expression for $F_{z\bar{z}}$ one can easily compute

$$2i < (F_r)^z{}_z >^{ab} = 2iG^{(r)z\bar{z}} < (F_r)_{\bar{z}z} >^{ab} = -\frac{4iU_2^{(r)}}{\mathcal{T}_2^{(r)}} (F_r)^{ab}_{z\bar{z}} = \frac{4\pi I_r}{\mathcal{T}_2^{(r)}} \tag{B.5}$$

and

$$2i < (F_r)^{\bar z}_{\bar z} >^{ab} = 2iG^{(r)\bar z z} < (F_r)_{z\bar z} >^{ab} = \frac{4iU_2^{(r)}}{\mathcal{T}_2^{(r)}}(F_r^{ab})_{z\bar z} = -\frac{4\pi I_r}{\mathcal{T}_2^{(r)}} \tag{B.6}$$

where the metric G^r given in eq. (A.3).

We also introduce

$$I^{ab} = I^a - I^b = -i\frac{U_2}{\pi}(F^a_{z\bar z} - F^b_{z\bar z})\,. \tag{B.7}$$

If one considers the quadratic terms in the action, there is no loss of generality in choosing the magnetization on the D brane labeled with the index b to be zero. This allows us to simplify the notation by writing $I^{ab} = I^a \equiv I$. We perform this choice in sections 2 and 3 while we will reintroduce the indexes when considering the Yukawa couplings in section 4.

Using the metric for the torus T^2 given in appendix A one gets:[14]

$$\tilde D_k \tilde D^k = \tilde D_k G^{ki}\tilde D_i = \begin{pmatrix} \tilde D_z & \tilde D_{\bar z}\end{pmatrix}\frac{2U_2}{\mathcal{T}_2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\tilde D_z \\ \tilde D_{\bar z}\end{pmatrix} = \frac{2U_2}{\mathcal{T}_2}\left\{\tilde D_z, \tilde D_{\bar z}\right\}. \tag{B.8}$$

If $I > 0$ we can introduce the creation and annihilation operator:

$$-i\tilde D_z \equiv -i\left(\partial_z - \frac{\pi I \bar z}{2U_2}\right) = \sqrt{\frac{\pi I}{U_2}}a^\dagger\,; \qquad -i\tilde D_{\bar z} \equiv -i\left(\partial_{\bar z} + \frac{\pi I z}{2U_2}\right) = \sqrt{\frac{\pi I}{U_2}}a \tag{B.9}$$

that satisfy the harmonic oscillator algebra:

$$[a, a^\dagger] = 1\,. \tag{B.10}$$

Using eqs. (B.9) in eq. (B.8) we get:

$$-\tilde D_k \tilde D^k = \frac{2\pi I}{\mathcal{T}_2}\left(aa^\dagger + a^\dagger a\right) = \frac{2\pi I}{\mathcal{T}_2}\left(2a^\dagger a + 1\right) \equiv \frac{2\pi I}{\mathcal{T}_2}\left(2N + 1\right). \tag{B.11}$$

The ground state for the torus T^2 is degenerate and there are I independent solutions given by:

$$\phi^{ab,n}_{T^2}(z) = e^{\pi i I z \frac{\mathrm{Im} z}{\mathrm{Im} U}}\,\Theta\begin{bmatrix}\frac{2n}{I}\\ 0\end{bmatrix}(Iz|IU)\,; \qquad n = 0 \ldots I-1 \tag{B.12}$$

which are determined by solving the equation

$$a\,\phi^{ab}_{T^2}(z,\bar z) \equiv \tilde D_{\bar z}\phi^{ab}_{T^2}(z,\bar z) = 0 \tag{B.13}$$

with the following periodicity conditions to be satisfied in going around the two one-cycles of the torus:

$$\phi^{ab}(z+1, \bar z+1) = e^{i\,\chi_1(z,\bar z)}\phi^{ab}(z,\,\bar z) \qquad \phi^{ab}(z+U, \bar z+\bar U) = e^{i\,\chi_2(z,\bar z)}\phi^{ab}(z,\,\bar z) \tag{B.14}$$

[14] In this equation and in the entire analysis of the torus T^2 the index k runs only on one torus and should not be confused with the one used in eq. (2.15).

where

$$\chi_1 = \frac{\pi I}{\mathrm{Im} U}\mathrm{Im}(z)\,; \qquad \chi_2 = \frac{\pi I}{\mathrm{Im} U}\mathrm{Im}(\bar U\, z)\,. \tag{B.15}$$

Remember that we use the following definition of the Θ-function:

$$\Theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|U) = \sum_n \mathrm{e}^{2\pi i\left[\frac{1}{2}(n+\frac{\alpha}{2})^2 U + (n+\frac{\alpha}{2})(z+\frac{\beta}{2})\right]}. \tag{B.16}$$

If $I < 0$ the identification of D_z and $D_{\bar z}$ with the creation and annihilation operators is exchanged; i.e.:

$$-\,i\tilde D_z = \sqrt{\frac{\pi|I|}{U_2}}a\ ; \qquad -\,i\tilde D_{\bar z} = \sqrt{\frac{\pi|I|}{U_2}}a^\dagger\,. \tag{B.17}$$

The operator in eq. (B.11) becomes:

$$-\,\tilde D_k \tilde D^k = \frac{2\pi|I|}{\mathcal{T}_2}\left(2a^\dagger a+1\right) \equiv \frac{2\pi|I|}{\mathcal{T}_2}\left(2N+1\right). \tag{B.18}$$

The wave functions of the (degenerate) ground state, are given by:

$$\phi^{ab,n}_{T^2} = e^{\pi i|I|\bar z\frac{\mathrm{Im}\bar z}{\mathrm{Im}\bar U}}\,\Theta\begin{bmatrix} \frac{-2n}{I} \\ 0 \end{bmatrix}(I\bar z|I\bar U)\,; \qquad n = 0\ldots|I|-1 \tag{B.19}$$

and are determined by requiring them to satisfy the following equation:

$$a\ \phi^{ab}_{T^2}(z,\bar z) \equiv \tilde D_z \phi^{ab}_{T^2}(z,\bar z) = 0 \tag{B.20}$$

and the periodicity conditions in eqs. (B.14). In particular, the structure of the phase factor in eq. (B.19) is fixed by the Laplace equation (B.20), while the arguments of the Theta function follow from the boundary conditions in eqs. (B.14) where we have used that eq. (B.15) can be equivalently written as

$$\chi_1 = \frac{\pi I}{\mathrm{Im}\bar U}\mathrm{Im}(\bar z)\,; \qquad \chi_2 = \frac{\pi I}{\mathrm{Im}\bar U}\mathrm{Im}(U\,\bar z)\,. \tag{B.21}$$

Eqs. (B.11) and (B.18) can be immediately generalized to the torus $T^2 \times T^2 \times T^2$ getting:

$$-\,\tilde D_k \tilde D^k \Longrightarrow \sum_{r=1}^{3} \frac{2U_2^{(r)}}{\mathcal{T}_2^{(r)}}\left\{\tilde D_{z_r}, \tilde D_{\bar z_r}\right\} = \sum_{r=1}^{3}\frac{2\pi|I_r|}{\mathcal{T}_2^{(r)}}\left(2N_r+1\right) \tag{B.22}$$

where the arrow indicates the change from the dimensional variables x^1, x^2 to the variables $z, \bar z$. In conclusion, eq. (2.15) (in dimensionless compact coordinates) can be written as:

$$-\,\tilde D_k \tilde D^k \phi^{ab}_n = m_n^2 \phi^{ab}_n \Longrightarrow \sum_{s=1}^{3}\frac{2\pi|I_s|}{\mathcal{T}_2^{(s)}}\left(2N_s+1\right)\phi^{ab}_n = \hat m_n^2 \phi^{ab}_n\,; \qquad m_n^2 = \frac{\hat m_n^2}{(2\pi R)^2}\,. \tag{B.23}$$

We go on in considering the eigenvalue equation for the fermions given in eq. (2.15). In particular, we restrict ourselves to the case $T^2\times T^2\times T^2$ and decompose the six-dimensional

Dirac algebra in the product of three two dimensional representations according to the relation:[15]

$$\begin{aligned} &\gamma^4_{(6)} = \gamma^1_{(1)} \otimes \sigma^3 \otimes \sigma^3\,; && \gamma^5_{(6)} = \gamma^2_{(1)} \otimes \sigma^3 \otimes \sigma^3 \\ &\gamma^6_{(6)} = \mathbb{I} \otimes \gamma^1_{(2)} \otimes \sigma^3\,; && \gamma^7_{(6)} = \mathbb{I} \otimes \gamma^2_{(2)} \otimes \sigma^3 \\ &\gamma^8_{(6)} = \mathbb{I} \otimes \mathbb{I} \otimes \gamma^1_{(3)}\,; && \gamma^9_{(6)} = \mathbb{I} \otimes \mathbb{I} \otimes \gamma^2_{(3)} \end{aligned} \tag{B.24}$$

with:

$$\gamma^2_{(r)} \equiv \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \gamma^1_{(r)} \equiv \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{B.25}$$

Correspondingly, the ten dimensional Majorana-Weyl spinors are the product of a four-dimensional spinor and three two-dimensional spinors $\eta_1 \otimes \eta_2 \otimes \eta_3$. The ten-dimensional Weyl condition imposes that these latter have to be Weyl spinors:

$$i\gamma^1_{(r)}\,\gamma^2_{(r)}\eta_r = \pm\eta_r\,. \tag{B.26}$$

The Dirac matrices, previously introduced, satisfy the Clifford algebra with a flat metric. On the torus T^2, in the complex coordinates, the metric is given in the second equation in (A.3), and therefore the flat Dirac matrices has to be multiplied by a suitable vierbein: i.e. $\gamma^i = e^i{}_I\gamma^I$ that is given in eq. (A.10). From it we get the Dirac matrices with a "curved" index:

$$\gamma^z_{(r)} = e^z{}_I\gamma^I_{(r)} = \sqrt{\frac{U_2^{(r)}}{T_2^{(r)}}}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}; \qquad \gamma^{\bar z}_{(r)} = e^{\bar z}{}_I\gamma^I_{(r)} = \sqrt{\frac{U_2^{(r)}}{T_2^{(r)}}}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \tag{B.27}$$

and therefore we can write:

$$\gamma^{z^r}_{(6)} = \mathbb{I}^{\otimes(r-1)} \otimes \gamma^z_{(r)} \otimes (\sigma^3)^{\otimes(3-r)}\,; \qquad \gamma^{\bar z^r}_{(6)} = \mathbb{I}^{\otimes(r-1)} \otimes \gamma^{\bar z}_{(r)} \otimes (\sigma^3)^{\otimes(3-r)} \tag{B.28}$$

where $V^{\otimes n} = V \otimes \cdots \otimes V$ with n V-factors.

Having defined the Dirac matrices, we go back to the eigenvalue equation in eq. (2.15) and we square it, getting:

$$\left(-\tilde D_i \tilde D^i \mathbb{I} - \frac{1}{2}[\gamma^i, \gamma^j]\tilde D_i \tilde D_j\right)\eta_n = \lambda_n^2\eta_n\,. \tag{B.29}$$

For a single torus the second term in the l.h.s. of the previous equation is given by:

$$\frac{1}{2}[\gamma^i,\gamma^j]\tilde D_i\tilde D_j = \frac{1}{2(2\pi R)^2}[\gamma^z,\gamma^{\bar z}][\tilde D_z, \tilde D_{\bar z}] = \frac{1}{(2\pi R)^2}\frac{2\pi I}{T_2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.30}$$

where eq. (B.3) has been used. Eqs. (B.30) and (B.11) we can put the fermionic eigenvalue equation in the following form:

$$\frac{1}{(2\pi R)^2}\left[2\pi\sum_{r=1}^{3}(2N_r+1)\frac{|I_r|}{T_2^{(r)}}\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} - \frac{2\pi I_1}{T_2^{(1)}}\sigma_3\otimes\mathbb{I}\otimes\mathbb{I} - \frac{2\pi I_2}{T_2^{(2)}}\mathbb{I}\otimes\sigma_3\otimes\mathbb{I}\right.$$
$$\left. - \frac{2\pi I_3}{T_2^{(3)}}\mathbb{I}\otimes\mathbb{I}\otimes\sigma_3\right]\eta_n^1\otimes\eta_n^2\otimes\eta_n^3 = \lambda_n^2\eta_n^1\otimes\eta_n^2\otimes\eta_n^3 \tag{B.31}$$

[15] See ref. [13].

where we have decomposed $\eta_n = \eta_n^1 \otimes \eta_n^2 \otimes \eta_n^3$. This equation shows that, for arbitrary signs of I_r ($r = 1, 2, 3$), there is always a unique zero mode that is a chiral fermion. In particular, if $I_{1,2,3}$ are all positive, then all three wave functions $\eta^{(1,2,3)}$ will have positive chirality: $\sigma_3 \eta = \eta$. Since the original ten-dimensional fermion is a Weyl fermion with chirality χ_{10}, the four-dimensional chirality χ_4 will be equal to $\chi_4 = \chi_{10}\chi_1\chi_2\chi_3$ where χ_r ($r = 1, 2, 3$) is the chirality on the r-th torus.

Since the zero mode eigenfunction on $T^2 \times T^2 \times T^2$ is the product of the zero mode eigenfunctions on each torus T^2, we will limit ourselves to the Dirac equation on the torus T^2:

$$\left(\gamma^z_{(r)} \tilde{D}_{z^r} + \gamma^{\bar{z}}_{(r)} \tilde{D}_{\bar{z}^r} \right) \eta_r^{ab}(z^r, \bar{z}^r) = 0 \tag{B.32}$$

where we have omitted the index 0 to simplify the notation, and it is satisfied when

$$\left(\gamma^z_{(r)} \tilde{D}_{z^r} + \gamma^{\bar{z}}_{(r)} \tilde{D}_{\bar{z}^r} \right) \eta_r^{ab}(z^z, \bar{z}^r) = 2\sqrt{\frac{U_2}{T_2}} \begin{pmatrix} 0 & \tilde{D}_{z^r} \\ \tilde{D}_{\bar{z}^r} & 0 \end{pmatrix} \begin{pmatrix} \eta^{ab}_{r,+} \\ \eta^{ab}_{r,-} \end{pmatrix} = 0 \tag{B.33}$$

with $\tilde{D}$ given in eq. (B.9). The Weyl condition written in eq. (B.26) imposes that the spinor has to be of the form:

$$\eta_{r,+} = \begin{pmatrix} \eta^{ab}_{r,+} \\ 0 \end{pmatrix}; \qquad \eta_{r,-} = \begin{pmatrix} 0 \\ \eta^{ab}_{r,-} \end{pmatrix} \tag{B.34}$$

with η_+ and η_- spinors with opposite chirality. By using again eq. (B.9), we have that the solution of eq. (B.33) imposes:

$$a_{(r)}\, \eta^{ab}_{r,+} \equiv -i\tilde{D}_{\bar{z}^r}\, \eta^{ab}_{r,+} = 0 \qquad I_r > 0 \tag{B.35}$$

while for $I_r < 0$, as previously discussed, the role of creation and annihilation operators is exchanged and we have:

$$a_{(r)}\, \eta^{ab}_{r,-} \equiv -i\tilde{D}_{z^r}\, \eta^{ab}_{r,-} = 0 \qquad I_r < 0\,. \tag{B.36}$$

The previous equations coincide with those for the bosonic degrees of freedom (eqs. (B.13) and (B.20)) and thus the solutions exactly coincide with the ones in eq. (2.25)

$$\eta^{ab,n_r}_{r,+} = \phi^{ab,n_r}_{r,+}; \qquad \eta^{ab,n_r}_{r,-} = \phi^{ab,n_r}_{r,-} \tag{B.37}$$

with $\eta^{ab}_{r,-} = (\eta^{ba}_{r,+})^\dagger$. In particular, if $I > 0$ ($I < 0$), then the spinor has positive (negative) chirality because the spinor with the opposite chirality has a wave-function that diverges for large values of Imz.

In the last part of this appendix we extend the previous analysis to the case of the fermions of the $\mathcal{N} = 2$ hypermultiplet. Such fermions appear in our model when one of the three tori, for example the third torus, is not magnetized, i.e. $I_3 = 0$. In this case, the equations of motion (B.32) for the lowest massless components of the mode expansion, are along the first two tori analogously to the $\mathcal{N} = 1$ case, while on the third torus the covariant derivative becomes the normal derivative. Also the boundary conditions are unchanged on the first two tori, while on the third one they become just the periodicity conditions of the wave-function when translated along the two one-cycles of T^2. These simple considerations allow us to immediately write down the compact wave functions of the massless hypermultiplet fermions. They coincide, along the first two tori, with the ones of the chiral fermions written for example in eq. (B.34), while are constant spinors along the third torus. In particular, the condition (B.26) implies that the constant spinor has to be a Weyl spinor. Depending on its chirality, we have two different solutions for the internal wave-function which have opposite six-dimensional chirality:

$$\eta_\alpha = \eta_{1,\pm} \otimes \eta_{2,\pm} \otimes \epsilon_\alpha \qquad \alpha = \{\uparrow, \downarrow\}\,; \qquad \epsilon_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \epsilon_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{B.38}$$

where we have suppressed the index labeling the mode expansion and the upper and lower signs can be independently chosen. The lowest massless fermionic state is now degenerate and, having both the ten-dimensional fermion and the internal wave-function a definite chirality, we have two four-dimensional fermions with opposite chirality. as expected. The full ten-dimensional wave-function can be written as follows:

$$\Psi_\alpha(x,y) = N_{\psi_\alpha}\, \psi_\alpha(x^\mu) \otimes \eta_\beta(y^i) \qquad \alpha, \beta = \{\uparrow, \downarrow\}\,. \tag{B.39}$$

C Evaluating the Yukawa couplings

In this appendix we give some details of the evaluation of the Yukawa couplings discussed in section 4 for the chiral multiplet. In particular, we will show how eqs. (4.5), (4.6) and (4.7) can be obtained starting from eq. (4.1) and considering only the zero modes. Let us concentrate our attention on the case in which the massless scalar is along the first torus (namely the condition in eq. (4.2)is satisfied). Eq. (4.1) becomes

$$\begin{aligned} S_3^\Phi \; = \; & \frac{1}{2g^2}\sqrt{\frac{\mathcal{T}_2^{(1)}}{2U_2^{(1)}}} \int d^4x \sqrt{G_4} \prod_{r=1}^{3} \left[(2\pi R)^2 \int d^2 z_r \sqrt{G^r} \right] \\ & \times \Big[\bar\psi^{ca}\, \gamma^5_{(4)} \varphi^{ab}_{1,-}\, \psi^{bc} \otimes (\eta^{ac})^\dagger \gamma^{z^1}_{(6)} \phi^{ab}_{1,+} \phi^{ab}_{2,\text{sign}\, I_2^{ab}} \phi^{ab}_{3,\text{sign}\, I_3^{ab}} \eta^{bc} \\ & \quad + \bar\psi^{ca}\, \gamma^5_{(4)} \varphi^{ab}_{1,+}\, \psi^{bc} \otimes (\eta^{ac})^\dagger \gamma^{\bar z^1}_{(6)} \phi^{ab}_{1,-} \phi^{ab}_{2,\text{sign}\, I_2^{ab}} \phi^{ab}_{3,\text{sign}\, I_3^{ab}} \eta^{bc} \\ & \quad - \bar\psi^{ca}\, \gamma^5_{(4)} \varphi^{bc}_{1,-}\, \psi^{ab} \otimes (\eta^{ac})^\dagger \gamma^{z^1}_{(6)} \phi^{bc}_{1,+} \phi^{bc}_{2,\text{sign}\, I_2^{bc}} \phi^{bc}_{3,\text{sign}\, I_3^{bc}} \eta^{ab} \\ & \quad - \bar\psi^{ca}\, \gamma^5_{(4)} \varphi^{bc}_{1,+}\, \psi^{ab} \otimes (\eta^{ac})^\dagger \gamma^{\bar z^1}_{(6)} \phi^{bc}_{1,-} \phi^{bc}_{2,\text{sign}\, I_2^{bc}} \phi^{bc}_{3,\text{sign}\, I_3^{bc}} \eta^{ab} \Big] \end{aligned} \tag{C.1}$$

where we have omitted the index 0 to simplify the notation and inserted the indexes a, b, c in order to dsitinguish the different brane magnetizations. According to the choice of the Chern classes signs, only one of the four terms in eq. (C.1) corresponds to the Yukawa coupling of the massless boson with the two fermions. Thus one has to compute one of the

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following integrals over the compact space

$$
\begin{aligned}
&\frac{1}{2g^2}\sqrt{\frac{\mathcal{T}_2^{(1)}}{2U_2^{(1)}}}\prod_{r=1}^{3}\left[(2\pi R)^2\int d^2z_r\sqrt{G^r}\right]\\
&\times\phi^{ab}_{1,+}(\eta_1^{ac\,\dagger}\gamma_1^z\eta_1^{bc})\otimes\phi^{ab}_{2,\mathrm{sign}\,I_2^{ab}}(\eta_2^{ac\,\dagger}\sigma_3\eta_2^{bc})\otimes\phi^{ab}_{3,\mathrm{sign}\,I_3^{ab}}(\eta_3^{ac\,\dagger}\sigma_3\eta_3^{bc})\\
=&\frac{1}{\sqrt{2}g^2}\prod_{r=1}^{3}\left[(2\pi R)^2\int d^2z_r\sqrt{G^r}\right]\\
&\times(\eta^{ca}_{1,-}\,\phi^{ab}_{1,+}\,\eta^{bc}_{1,-})(\mathrm{sign}\,I_2^{ac}\eta^{ca}_{2,\mp}\,\phi^{ab}_{2,\mathrm{sign}\,I_2^{ab}}\,\eta^{bc}_{2,\pm})(\mathrm{sign}\,I_3^{ac}\eta^{ca}_{3,\mp}\,\phi^{ab}_{3,\mathrm{sign}\,I_3^{ab}}\,\eta^{bc}_{3,\pm})
\end{aligned}
\tag{C.2}
$$

for the case $I_1^{ab}>0$ and $I_1^{bc},I_1^{ca}<0$,

$$
\begin{aligned}
&\frac{1}{2g^2}\sqrt{\frac{\mathcal{T}_2^{(1)}}{2U_2^{(1)}}}\prod_{r=1}^{3}\left[(2\pi R)^2\int d^2z_r\sqrt{G^r}\right]\\
&\times\phi^{ab}_{1,-}(\eta_1^{ac\,\dagger}\gamma_1^{\bar z}\eta_1^{bc})\otimes\phi^{ab}_{2,\mathrm{sign}\,I_2^{ab}}(\eta_2^{ac\,\dagger}\sigma_3\eta_2^{bc})\otimes\phi^{ab}_{3,\mathrm{sign}\,I_3^{ab}}(\eta_3^{ac\,\dagger}\sigma_3\eta_3^{bc})\\
=&\frac{1}{\sqrt{2}g^2}\prod_{r=1}^{3}\left[(2\pi R)^2\int d^2z_r\sqrt{G^r}\right]\\
&\times(\eta^{ca}_{1,+}\,\phi^{ab}_{1,-}\,\eta^{bc}_{1,+})(\mathrm{sign}\,I_2^{ac}\eta^{ca}_{2,\mp}\,\phi^{ab}_{2,\mathrm{sign}\,I_2^{ab}}\,\eta^{bc}_{2,\pm})(\mathrm{sign}\,I_3^{ac}\eta^{ca}_{3,\mp}\,\phi^{ab}_{3,\mathrm{sign}\,I_3^{ab}}\,\eta^{bc}_{3,\pm})
\end{aligned}
\tag{C.3}
$$

for the case $I_1^{ab}<0$ and $I_1^{bc},I_1^{ca}>0$,

$$
\begin{aligned}
&-\frac{1}{\sqrt{2}g^2}\prod_{r=1}^{3}\left[(2\pi R)^2\int d^2z_r\sqrt{G^r}\right]\\
&\times(\eta^{ca}_{1,-}\,\phi^{bc}_{1,+}\,\eta^{ab}_{1,-})(\mathrm{sign}\,I_2^{ac}\eta^{ca}_{2,\mp}\,\phi^{bc}_{2,\mathrm{sign}\,I_2^{bc}}\,\eta^{ab}_{2,\pm})(\mathrm{sign}\,I_3^{ac}\eta^{ca}_{3,\mp}\,\phi^{bc}_{3,\mathrm{sign}\,I_3^{bc}}\,\eta^{ab}_{3,\pm})
\end{aligned}
\tag{C.4}
$$

for $I_1^{bc}>0$ and $I_1^{ab},I_1^{ca}<0$, and

$$
\begin{aligned}
&-\frac{1}{\sqrt{2}g^2}\prod_{r=1}^{3}\left[(2\pi R)^2\int d^2z_r\sqrt{G^r}\right]\\
&\times(\eta^{ca}_{1,+}\,\phi^{bc}_{1,-}\,\eta^{ab}_{1,+})(\mathrm{sign}\,I_2^{ac}\eta^{ca}_{2,\mp}\,\phi^{bc}_{2,\mathrm{sign}\,I_2^{bc}}\,\eta^{ab}_{2,\pm})(\mathrm{sign}\,I_3^{ac}\eta^{ca}_{3,\mp}\,\phi^{bc}_{3,\mathrm{sign}\,I_3^{bc}}\,\eta^{ab}_{3,\pm})
\end{aligned}
\tag{C.5}
$$

for $I_1^{bc}<0$ and $I_1^{ab},I_1^{ca}>0$. The cases in which $I_1^{ca}>0$ and $I_1^{ab},I_1^{bc}<0$ and the one in which $I_1^{ca}<0$ and $I_1^{ab},I_1^{bc}>0$ can be obtained from eqs. (C.4) and (C.5) respectively, by changing the indices ca with bc. Notice that, in order to get a non-vanishing expression, the two fermionic components of the internal wave function along the second and the third torus need to have opposite chiralites.

With the choice of the sign for the Chern classes along the first torus given in eq. (4.4), the internal wave function associated with the bosonic zero mode solution is the first one in eq. (2.25), and thus only the first term in eq. (C.1) contributes. In this case to determine the Yukawa coupling one has to evaluate the integral in eq. (C.2).

in order to write a general expression of the Yukawa couplings which holds for each of the previous choices of the Chern classes signs, one can introduce a factor σ =

$\mathrm{sign}(I_1^{bc} I_1^{ca} I_2^{ac} I_3^{ac}) = \pm 1$ which encodes the signs $\mathrm{sign}(I_2^{ac})$ and $\mathrm{sign}(I_3^{ac})$, relative to the second and the third torus and the overall sign of the coupling which is $+$ in the case of eqs. (C.2)–(C.3) and $-$ in the case of eqs. (C.4)–(C.5).

Substituting eqs. (2.25)) in the previous expression, we end up with the following product of overlap integrals of three Θ-functions

$$Y = \frac{1}{g^2}\sigma \prod_{r=1}^{3} \int_{T_r^2} d^2 z_r \sqrt{G^r} \phi^{ab,l_r}_{\mathrm{sign}\, I^{ab}_r} \phi^{ca,n_r}_{\mathrm{sign}\, I^{ca}_r} \phi^{bc,m_r}_{\mathrm{sign}\, I^{bc}_r} . \tag{C.6}$$

Let us first restrict ourselves to the case of the first torus T^2. In order to calculate the previous integral one has to use the addition formula for the Θ-functions [14]

$$\begin{aligned}\Theta\begin{bmatrix}\frac{2a}{n_1}\\0\end{bmatrix}(z_1|n_1\Omega)\Theta\begin{bmatrix}\frac{2b}{n_2}\\0\end{bmatrix}(z_2|n_2\Omega) = &\sum_{d\in Z_{(n_1+n_2)}} \Theta\begin{bmatrix}\frac{2(n_1 d+a+b)}{n_1+n_2}\\0\end{bmatrix}(z_1+z_2|(n_1+n_2)\Omega) \\ &\times\Theta\begin{bmatrix}\frac{2(n_1 n_2 d+n_2 a-n_1 b)}{n_1 n_2(n_1+n_2)}\\0\end{bmatrix}(n_2 z_1 - n_1 z_2|n_1 n_2(n_1+n_2)\Omega)\end{aligned} \tag{C.7}$$

where $Z_{(n_1+n_2)}$ indicates the set of the integer numbers modulo (n_1+n_2). In our example, being $n_1 \equiv I^{ca}, n_2 \equiv I^{bc} < 0$ and $I^{ab} > 0$ with $I^{bc}+I^{ca}+I^{ab} = 0$, we have

$$\begin{aligned}\phi_-^{ca,n}(\bar z)\phi_-^{bc,m}(\bar z) = e^{i\pi I^{ab}\bar z \frac{\mathrm{Im}\bar z}{\mathrm{Im}U}} &\sum_{d\in \mathbb{Z}_{I^{ca}+I^{bc}}} \Theta\begin{bmatrix}\frac{2(dI^{ca}-n-m)}{I^{ba}}\\0\end{bmatrix}(I^{ba}\bar z|I^{ba}\bar U) \\ &\times\Theta\begin{bmatrix}\frac{2(dI^{bc}I^{ca}-nI^{bc}+mI^{ca})}{I^{ca}I^{bc}I^{ba}}\\0\end{bmatrix}(0|I^{bc}I^{ca}I^{ba}\bar U).\end{aligned} \tag{C.8}$$

Let us focus on the terms that depend on z and $\bar z$ and leave for a moment aside the last term in eq. (C.8). These terms in fact contribute to the integral in eq. (C.6) on the first torus T^2. For each value of the index d in the sum in eq. (C.8), one has to evaluate the following integral:

$$\int d^2 z e^{i\pi I^{ab}\bar z\frac{\mathrm{Im}\bar z}{\mathrm{Im}U}}\Theta\begin{bmatrix}\frac{2(dI^{ac}+m+n)}{I^{ab}}\\0\end{bmatrix}(I^{ba}\bar z|I^{ba}\bar U) e^{i\pi I^{ab} z \frac{\mathrm{Im}z}{\mathrm{Im}U}}\Theta\begin{bmatrix}\frac{2l}{I^{ab}}\\0\end{bmatrix}(I^{ab}z|I^{ab}U)\,. \tag{C.9}$$

By defining:

$$z \equiv x + Uy \ ; \qquad 0\le x \le 1; \qquad 0 \le y \le 1; \qquad U \equiv U_1 + iU_2 \tag{C.10}$$

the previous integral becomes:

$$\begin{aligned}&T_2 \int_0^1 dx \int_0^1 dy e^{i\pi I^{ab}(U-\bar U)y^2} \sum_{q,q'=-\infty}^{\infty} e^{i\pi I^{ab}\left[\left(q'+\frac{l}{I^{ab}}\right)^2 U - \left(q+\frac{dI^{ac}+m+n}{I^{ab}}\right)^2\bar U\right]} \\ &\times e^{2\pi i I^{ab}\left[\left(q'+\frac{l}{I^{ab}}\right)-\left(q+\frac{dI^{ac}+m+n}{I^{ab}}\right)\right]x}\, e^{2i\pi I^{ab}\left[\left(q'+\frac{l}{I^{ab}}\right)U - \left(q+\frac{dI^{ac}+m+n}{I^{ab}}\right)\bar U\right]y}.\end{aligned} \tag{C.11}$$

The integral over x can be easily performed and, in order to get a non-zero result, one has to impose the following relation:

$$q + \frac{dI^{ac} + m + n}{I^{ab}} = q' + \frac{l}{I^{ab}} . \tag{C.12}$$

The integral over y can be rewritten as follows:

$$\int_0^1 dy e^{-2\pi I^{ab} U_2 y^2} \sum_{q'} e^{-2\pi I^{ab}\left(q'+\frac{l}{I^{ab}}\right)^2 U_2} e^{-4\pi I^{ab}\left(q'+\frac{l}{I^{ab}}\right) U_2 y} = \int_0^1 dy \sum_{q'} e^{-2\pi I^{ab} U_2 \left(y+q'+\frac{l}{I^{ab}}\right)^2} .$$

In conclusion, eq. (C.11) is equal to:

$$T_2 \int_0^1 du \sum_{q'=-\infty}^{\infty} e^{-2\pi I^{ab} U_2 \left(u+q'+\frac{l}{I^{ab}}\right)^2} = T_2 \int_{-\infty}^{\infty} du e^{-2\pi I^{ab} U_2 \left(u+\frac{l}{I^{ab}}\right)^2} = \frac{T_2}{(2I^{ab}U_2)^{1/2}} \tag{C.13}$$

where we have used the identity:

$$\int_0^1 du \sum_{n=-\infty}^{\infty} F(n+u) = \int_{-\infty}^{\infty} du F(u) \tag{C.14}$$

which trivially follows from:

$$\int_0^1 du \sum_{n=-\infty}^{\infty} F(n+u) = \lim_{A\to\infty} \sum_{n=-A}^{A} \int_n^{n+1} dx F(x) = \lim_{A\to\infty} \int_{-A}^{A} dx F(x) . \tag{C.15}$$

Finally, one gets the following result for the integral in eq. (C.9):

$$\begin{aligned} &\int d^2 z e^{i\pi I^{ab} \bar{z} \frac{\mathrm{Im}\bar{z}}{\mathrm{Im}U}} \Theta \begin{bmatrix} \frac{2(dI^{ac}+m+n)}{l} \\ 0 \end{bmatrix} (I^{ba}\bar{z}|I^{ba}\bar{U}) e^{i\pi I^{ab} z \frac{\mathrm{Im}z}{\mathrm{Im}U}} \Theta \begin{bmatrix} \frac{2l}{I^{ab}} \\ 0 \end{bmatrix} (I^{ab}\bar{z}|I^{ab}\bar{U}) \\ &= \frac{T_2}{(2I^{ab}U_2)^{1/2}} \delta_{dI^{ac}+m+n;l} . \end{aligned} \tag{C.16}$$

Here the δ-function comes from the integration over x, that gives a non-vanishing result only if

$$dI^{ac} + m + n - l = kI^{ab} \tag{C.17}$$

which can be equivalently written as

$$dI^{ac} + m + n - l = 0 \tag{C.18}$$

using that the integer l is defined modulus I^{ab}. On the other hand, as one can see from eq. (C.8), d is an integer modulus I^{ba}. Therefore the result on the integration on the first torus T^2 is non vanishing only if, given the integers m, n, l, a value of d in the interval $0 \le d \le I^{ab} - 1$ can be found such that the quantity in eq. (C.18) is an integer. Then, including the constant term in z that appears in the second line of eq. (C.8) and assuming that there is a value of d such that the quantity in eq. (C.18) is an integer, one gets the contribution to the Yukawa coupling coming from the first torus T^2:

$$Y = \frac{\sigma}{g^2}\frac{T_2}{(2I^{ab}U_2)^{1/2}}\Theta\begin{bmatrix}\frac{2m}{I^{ca}I^{bc}} - \frac{2l}{I^{ca}I^{ba}} \\ 0\end{bmatrix}(0|I^{bc}I^{ca}I^{ba}\bar{U})\,. \tag{C.19}$$

The characteristc of the Θ function can be written in a more general way, which is valid for each value of the Chern classess, as follows [8]:

$$\begin{aligned}\frac{2m}{I^{ca}I^{bc}} - \frac{2l}{I^{ca}I^{ba}} &= \frac{2}{I^{ca}}\left(\frac{m}{I^{bc}} + \frac{l}{I^{ab}}\right) = \frac{2}{I^{ca}}\left(\frac{m'I^{ab}}{I^{bc}} + \frac{l'I^{bc}}{I^{ab}}\right) \\ &= -2\left\{\frac{m'+l'}{I^{ca}} + \frac{m'}{I^{bc}} + \frac{l'}{I^{ab}}\right\} = -2\left\{\frac{n'}{I^{ca}} + \frac{m'}{I^{bc}} + \frac{l'}{I^{ab}}\right\}\end{aligned} \tag{C.20}$$

where we have made a ridefinition of the indices $m, l \to m', l'$ which are still defined modulus I^{bc} and modulus I^{ab} respectively. Such a redefinition is allowed for (I^{bc}, I^{ab}, I^{ca}) relative prime. Moreover, we have introduced $n' = m' + l'$ which is defined modulus I^{ca}. Then substituting eq. (C.20) in Eq (C.19) one gets

$$Y = \frac{\sigma}{g^2}\,\frac{T_2}{(2I^{ab}U_2)^{1/2}}\Theta\begin{bmatrix}2\left(\frac{n'}{I^{ca}} + \frac{m'}{I^{bc}} + \frac{l'}{I^{ab}}\right) \\ 0\end{bmatrix}(0|I^{bc}I^{ca}I^{ba}\bar{U}) \tag{C.21}$$

where we have omitted the minus sign in the characteristic of the Θ-function and used the property

$$\Theta\begin{bmatrix}-a \\ 0\end{bmatrix}(0|t) = \Theta\begin{bmatrix}a \\ 0\end{bmatrix}(0|t)\,.$$

In order to generalize the previous result to the case of the torus $T^2 \times T^2 \times T^2$ (always performing the choice in eq. (4.4)), we notice from eq. (C.2) that the following integral has to be computed both along the second and the third torus:

$$\int_{T^2_r} d^2 z_r \sqrt{G^r}\,\phi^{ca}_{r,\mp}\,\phi^{ab}_{r,\mathrm{sign}\,I^{ab}}\phi^{bc}_{r,\pm}\,. \tag{C.22}$$

One has to apply again the addition formula (C.8) to ϕ^{ab}_r and ϕ^{ca}_r if $\mathrm{sign}(I^{ab}_r I^{ca}_r) > 0$ or to ϕ^{ab}_r and ϕ^{bc}_r if $\mathrm{sign}(I^{ab}_r I^{bc}_r) > 0$. Following the same calculations done in eqs. (C.8)–(C.13) one ends with a normalization factor

$$\begin{aligned}&\frac{T_2^{(r)}}{(2|I^{bc}_r|U_2^{(r)})^{1/2}} \to \text{for}\quad \mathrm{sign}(I^{ab}_r I^{ca}_r) > 0 \\ &\frac{T_2^{(r)}}{(2|I^{ca}_r|U_2^{(r)})^{1/2}} \to \text{for}\quad \mathrm{sign}(I^{ab}_r I^{bc}_r) > 0\,.\end{aligned} \tag{C.23}$$

Notice that with a choice different from the one in (4.4), for instance $I^{ab}_1, I^{ca}_1 < 0$ and $I^{bc}_1 > 0$), one should start from eq. (C.4) rather than eq. (C.2) and thus the integral to compute along the second and the third torus would be

$$\int_{T^2_r} d^2 z_r \sqrt{G^r}\,\phi^{ca}_{r,\mp}\,\phi^{bc}_{r,\mathrm{sign}\,I^{bc}}\phi^{ab}_{r,\pm} \tag{C.24}$$

and, instead of eq. (C.23), one would find:

$$\begin{aligned}
&\frac{T_2^{(r)}}{(2|I_r^{ab}|U_2^{(r)})^{1/2}} \to \text{for} \quad \text{sign}(I_r^{bc} I_r^{ca}) > 0 \\
&\frac{T_2^{(r)}}{(2|I_r^{ca}|U_2^{(r)})^{1/2}} \to \text{for} \quad \text{sign}(I_r^{ab} I_r^{bc}) > 0\,.
\end{aligned} \tag{C.25}$$

Defining

$$\begin{aligned}
\chi_r^{ab} &= (1+\text{sign}(I_r^{bc} I_r^{ca}))/2 \\
\chi_r^{bc} &= (1+\text{sign}(I_r^{ab} I_r^{ca}))/2 \\
\chi_r^{ca} &= (1+\text{sign}(I_r^{bc} I_r^{ab}))/2
\end{aligned} \tag{C.26}$$

one can write the two previous results in a unified way as

$$\frac{T_2^{(r)}}{\left(2U_2^{(r)}|I_r^{bc}|\chi_r^{bc}|I_r^{ab}|\chi_r^{ab}|I_r^{ca}|\chi_r^{ca}\right)^{1/2}} \tag{C.27}$$

for all the three tori.

D Supersymmetry transformations

The action of the ten-dimensional $\mathcal{N}=1$ super Yang-Mills, written in eq. (2.1), is invariant under the global supersymmetric variations [15]:

$$\delta A_M = \frac{i}{2}\,\bar\epsilon\,\Gamma_M\,\lambda\,; \qquad \delta\lambda = -\frac{1}{4}\,\Gamma^{MN}\,F_{MN}\,\epsilon \tag{D.1}$$

where ϵ is a ten-dimensional constant spinor. Starting from the ten dimensional supersymmetry transformations and performing the Kaluza-Klein reduction we can determine the four-dimensional supersymmetries which are preserved in our flux compactification. This analysis can be carried in both the twisted and untwisted sectors. In the following we restrict our attention only to the twisted sector. In particular, by implementing in eq. (D.1) the decompositions written in eq. (2.4), restricting our analysis only to the massless fields in the bifundamental representation of the gauge group $\mathrm{U}(1)_a\times\mathrm{U}(1)_b$, using eq. (2.6) and the mode expansions given in eq. (2.14), we can write:

$$N_{\varphi_1}\,\delta[\varphi^{ab}_{z^1}(x)\otimes\phi^{ab}_0(y)] = i\frac{N_\psi}{2}\bar\epsilon_4\gamma^5_{(4)}\psi^{ab}(x)\otimes G_{z^1\bar z^1}\epsilon_1^\dagger\gamma^{\bar z^1}\eta_1^{ab}\otimes\epsilon_2^\dagger\;\mathbb{I}\;\eta_2\otimes\epsilon_3\;\mathbb{I}\;\eta^{ab}. \tag{D.2}$$

Here, N_1 and N_ψ are the normalization factors that we have introduced in order to have four dimensional actions with the correct holomorphic properties, as extensively discussed in this paper.

Eq. (D.2) has been obtained by decomposing the ten-dimensional spinor ϵ as a product of a four-dimensional spinor and three two-dimensional ones as follows:

$$\epsilon = \epsilon_{(4)}\otimes\begin{pmatrix}\epsilon_1^+\\ \epsilon_1^-\end{pmatrix}\otimes\begin{pmatrix}\epsilon_2^+\\ \epsilon_2^-\end{pmatrix}\otimes\begin{pmatrix}\epsilon_3^+\\ \epsilon_3^-\end{pmatrix} \tag{D.3}$$

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in complete analogy with what we have done for the fermionic field.

The scalar involved in eq. (D.2) is massless when the constraint written in eq. (2.20) is satisfied for $r = 1$ and with $I_1^{ab} > 0$. In the following we assume that both these conditions are satisfied. This means that the internal total wave-function of the scalar is:

$$\phi_0^{ab} = \phi_{1,+}^{ab;\,n^1} \prod_{r=2}^{3} \phi_{r,\,\mathrm{sign}(I_r^{ab})}^{ab,\,n^r} \tag{D.4}$$

where the wave-functions for each torus are given in eq. (2.25). Analogously, the internal two-dimensional spinors η_r (r=1,2,3), which come from the decomposition of the six dimensional spinor η, have, according to the eq. (B.34), positive chirality on the first torus and positive or negative chirality on the other two tori, depending on the sign of I_r^{ab} for $r = 2, 3$. These considerations allow us to write the susy transformation, in the following way:

$$N_{\varphi_1}\delta[\varphi_{z^1}^{ab}(x) \otimes \phi_0^{ab}(y)] = i\frac{N_\psi}{2}\sqrt{\frac{\mathcal{T}_2^{(1)}}{U_2^{(1)}}}\bar{\epsilon}_4\gamma_{(4)}^5\psi^{ab}(x)\ (\epsilon_1^-\ \eta_{1,+}^{ab;\,n^1})\ (\epsilon_2^{\mathrm{sign}(I_2^{ab})}\eta_{2,\,\mathrm{sign}(I_2^{ab})}^{ab,\,n^2})$$
$$\times(\epsilon_3^{\mathrm{sign}(I_3^{ab})}\eta_{3,\mathrm{sign}(I_3^{ab})}^{ab,\,n^3}) \tag{D.5}$$

where the γ-matrix written in eq. (B.27) has been used. This equation shows that, in order to have a non-vanishing expression, the constant two-dimensional spinors have to be taken equal to: $\epsilon_1^- = 1$, $\epsilon_{2,3}^{\mathrm{sign}(I^{ab})} = 1$ with all the other components being zero. With this choice and remembering the relation between the bosonic and fermionic wave function, written in equation (B.37), we have:

$$\phi_0^{ab} = \eta_{1,+}^{ab;\,n^1}\ \eta_{2,\,\mathrm{sign}(I_2^{ab})}^{ab;\,n^2}\ \eta_{3,\mathrm{sign}(I_3^{ab})}^{ab;\,n^3} \tag{D.6}$$

where now the η's are the non-zero components of the chiral two-dimensional spinor. Introducing the scalar field φ_1 and the relation $N_{\varphi_1} = N_\psi/\sqrt{2\pi}$, both already defined in section 3, we can write the four-dimensional supersymmetric variation for the twisted fields as follows:

$$\delta\varphi_1^{ab} = \frac{i}{2}\bar{\epsilon}_4\gamma_{(4)}^5\psi^{ab}(x) \tag{D.7}$$

which explicitly shows that the supersymmetric partner of the fermion ψ is the field φ_1.

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